

Renormalization Group Studies of Vertex Models

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Chapter 1

Introduction

In this thesis we use renormalization group methods to study the critical behaviour of the staggered F-model. The staggered F-model, defined in chapter 2, can be used as a model of a facet of a BCC crystal in the (100) direction.

We first use known exact results to map the staggered F-model to a sine-Gordon type model (defined in section 3.1), and study the renormalization group equations for this model using momentum shell integration techniques. The map to the sine-Gordon model is constructed using an exact result on the long range part of the height-height correlation function of the F-model (i.e. the staggered F-model at zero staggered field) [40]. The results we obtain are the phase diagram of the staggered F-model and the leading singularity in the free energy.

To get more results, e.g. the next to leading singularity in the free energy, we need a larger source of information than is available in the form of the long range part of the height-height correlation function. It turns out that the free fermion method, upon which Baxter's original solution is based, is flexible enough to admit a perturbative expansion about the free fermion line. By calculating the singular part of the free energy perturbatively about the free fermion line, one can construct a map from the staggered F-model to the sine-Gordon model by demanding that it correctly reproduces the singular part of the free energy. The idea thus is to use the mapping to the sine-Gordon type model as an extrapolation technique. To make this approach practical we:

1. Develop in chapter 3 a simple diagrammatic method to find the renormalization group equations for a given sine-Gordon type model. This method is based on a combination of functional Feynman rules and the operator product expansion.
2. Rewrite in section 5.9 the perturbative expansion about the free fermion line as a linked cluster expansion.

Although these two results make it possible to construct a map to a sine-Gordon type model in a systematic way, the actual construction of this map is beyond the scope of this thesis. We do, however, explicitly calculate the first order correction to Baxter's result. This allows us to verify results which previously could only be obtained using renormalization group arguments.

1.1 Summary

In chapter 2 we introduce the staggered F-model and discuss the equivalence with the BCSOS model.

In chapter 3 a systematic method is derived to generate the renormalization group equations

In chapter 4 we first discuss the application of the renormalization group in the calculation of critical exponents. We then proceed to obtain the phase diagram of the staggered F-model and calculate the leading singularity in the free energy by using the information present in the form of the asymptotic form of the height-height correlation function

In chapter 5 we first present a derivation of Baxter's exact solution. We then perturbatively lift the free fermion condition. This allows one to write the free energy of the staggered F-model as a perturbative expansion about the free fermion line. Baxter's exact solution can thus be seen as the zeroth order term in this expansion. We explicitly calculate the first order term. To facilitate the computation of the higher order terms we derive a linked cluster method in section 5.9.

Chapter 2

Definition of the staggered F-model

In this chapter we shall introduce the six-vertex model, of which the staggered F-model is a special case. Some known results are discussed.

2.1 The six-vertex model

The six-vertex model can be defined as follows: place arrows on the bonds of a square lattice so that there are two arrows pointing into each vertex. Six types of vertices can arise (hence the name of the model). These vertices are shown in fig. 2.1. By giving each vertex-type a (position-dependent) energy the model is defined. These models were first introduced to study ferroelectric systems. Later it was shown that six-vertex models can be mapped to solid-on-solid (SOS) models [6]. Only a few of these models can be solved exactly. These include the free fermion models [8,39] and models that can be solved using a (generalized) Bethe Ansatz [4,5,21–23]. To define the staggered F-model, we divide the lattice into two sublattices A and B, such that the nearest neighbor of an A vertex is a B vertex. The vertex energies are chosen as indicated in fig. 2.1. When the staggered field (s) vanishes the model reduces to the F-model, which has been solved by Lieb [22]. For nonzero staggered field the model can be solved when $\beta\epsilon = \frac{1}{2} \ln(2)$ [3].

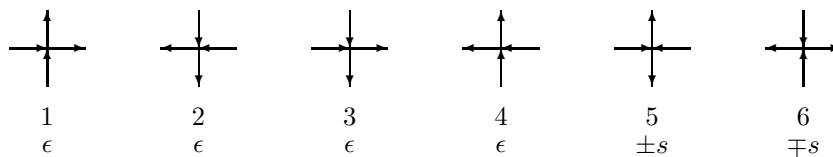


Figure 2.1: The six vertices and their energies. The upper and lower signs correspond to the two sublattices.

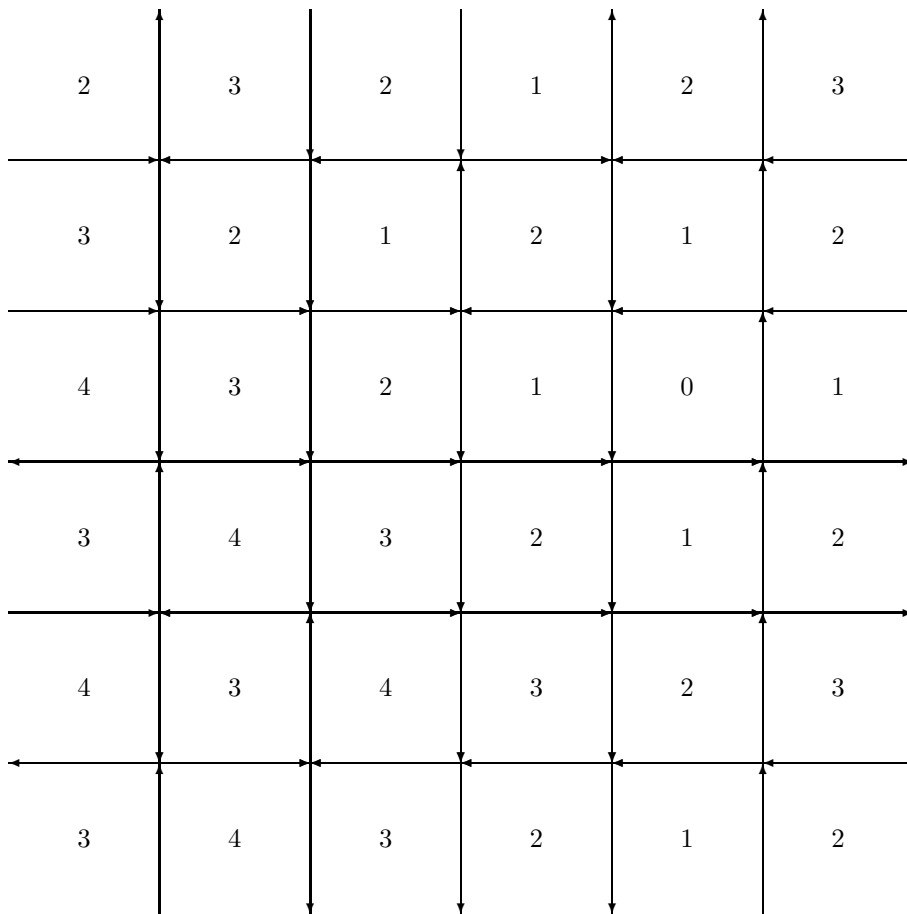


Figure 2.2: An arrow configuration together with the corresponding height function.

2.2 Six-vertex models and SOS models

We now proceed to show how six-vertex models are related to SOS models. First we introduce a dual lattice. Each bond of the dual lattice now crosses an arrow placed on one of the bonds of the original lattice. By rotating this arrow 90° clockwise and placing it on the corresponding bond of the dual lattice, we obtain an arrow configuration on the dual lattice. A height function (h) is now defined by demanding that $h(x) = h(y) + 1$ if an arrow points from y to x . By fixing the height at one particular point, the height at each point of the dual lattice is defined unambiguously. See [6] for more details. The fact that the height difference between nearest neighbors is always ± 1 makes six-vertex models ideal models for crystal surfaces of BCC crystals in the (100) direction. The class of SOS models to which six-vertex models are mapped is also known as body centered solid on solid models (BCSOS models). In fig. 2.2 an arrow configuration on a lattice together with the corresponding height function on the dual lattice is shown.

2.3 Roughening transition in the F-model

According to [22] a phase transition of Kosterlitz-Thouless type takes place in the F-model at inverse temperature $\beta\epsilon = \ln(2)$. If $\beta\epsilon > \ln(2)$ the crystal surface as described by the F-model is smooth. In this case the height-height correlation function $G(r) = \langle (h(r) - h(0))^2 \rangle$ decays exponentially with increasing r . When one takes $\beta\epsilon < \ln(2)$, the surface is in a rough phase. It can be shown that [40]

$$G(r) = \frac{2}{\pi \arccos(1 - \frac{1}{2} \exp(2\beta\epsilon))} \ln(r) \quad (2.1)$$

The logarithmic divergence of the correlation function at large distances is caused by thermal fluctuations in the local height of the surface with arbitrary long wavelengths. Note that for $\epsilon > 0$ the F-model has a twofold degenerate ground state consisting of vertex 5 on one sublattice and vertex 6 on the other sublattice. By introducing a staggered field this degeneracy is lifted. It has been shown [27] that in a nonzero staggered field the F-model is in a smooth phase for positive ϵ .

Chapter 3

Renormalization group equations for sine-Gordon type models

In this chapter we will introduce the sine-Gordon type Hamiltonian and then show how renormalization group equations can be obtained for such models. First a cut-off procedure will be introduced to define the theory. Renormalization is carried out by first integrating over some of the degrees of freedom of the model. The model, when formulated in terms of the remaining degrees of freedom, will look like the original model with a lower cut-off. Finally a scale transformation will restore the original cut-off.

3.1 Effective Hamiltonians for the staggered F-model

Since the staggered F-model can be interpreted as a solid-on-solid model (see section 2.2), it is natural to introduce a field h , that describes the height of a surface. The Hamiltonian density of this field must possess the same symmetries as the staggered F-model. In particular we must have:

$$F(h+1, \beta s) = F(h, -\beta s) \quad (3.1)$$

$$F(h) = F(-h) \quad (3.2)$$

Here s is the staggered field, and we have assumed that the ground state of the staggered F-model (for $\beta\epsilon > 0$ and $\beta s \neq 0$) corresponds to $h = 0$ in the sine-Gordon model. From (3.1) it follows that

$$F(h+2, \beta s) = F(h, \beta s) \quad (3.3)$$

(3.2) and (3.3) lead us to the Hamiltonian density:

$$F(h, \partial_i h, \partial_{ij} h, \dots, \beta\epsilon, \beta s) = \sum_{n=0}^{\infty} D_n(\partial_i h, \partial_{ij} h, \dots, \beta\epsilon, \beta s) \cos(n\pi h) \quad (3.4)$$

Here D_n is an unknown function of its arguments. According to (3.1) we have

$$D_n(\partial_i h, \partial_{ij} h, \dots, \beta\epsilon, -\beta s) = (-1)^n D_n(\partial_i h, \partial_{ij} h, \dots, \beta\epsilon, \beta s) \quad (3.5)$$

3.2 The renormalization group transformation

We will rewrite the Hamiltonian (3.4) as

$$H = \sum_{-\infty}^{\infty} \int \frac{d^2x}{a^2} \exp(in\pi h) D_n(a\partial_i h, a^2\partial_{ij} h, a^3\partial_{ijk} h, \dots) \quad (3.6)$$

We can think of the constant a as the “lattice constant” of the original microscopic Hamiltonian. In this original model $\frac{1}{a^2}$ would be the density of degrees of freedom. The effective Hamiltonian (3.6) should have the same density of degrees of freedom. The constant a appears in the Hamiltonian as a consequence of replacing summations by integrals and finite differences by partial derivatives. We will define the Fourier transform of the field $h(x)$ as

$$\hat{h}(k) = \frac{1}{\sqrt{V}} \int d^2x h(x) \exp(-ik \cdot x) \quad (3.7)$$

Here V is the volume of the system. $h(x)$ can then be written as

$$h(x) = \frac{1}{\sqrt{V}} \sum_k \hat{h}(k) \exp(ik \cdot x) \quad (3.8)$$

We now define a cut-off by introducing a set (S) of allowed k -values. We assume that the set S has the property:

$$k \in S \Rightarrow -k \in S \quad (3.9)$$

The density of k -values is written as $\frac{V}{(2\pi)^2} P(k)$. The function $P(k)$ will be called a cut-off function. We shall assume that the cut-off is chosen such that $P(0) = 1$ and all derivatives of $P(k)$ are zero at $k = 0$. If the volume V is chosen large enough, a summation over S can be replaced by an integral:

$$\sum_{k \in S} F(k) = V \int \frac{d^2k}{(2\pi)^2} P(k) F(k) \quad (3.10)$$

provided that the function F does not correlate with the characteristic function of S . In case such correlations do exist we have to replace $P(k)$ by the characteristic function of the set S , which we denote as $P_c(k)$. In general we thus have

$$\sum_{k \in S} F(k) = V \int \frac{d^2k}{(2\pi)^2} P_c(k) F(k) \quad (3.11)$$

The value of a now follows by requiring $\frac{1}{a^2}$ to be the number of degrees of freedom per unit volume:

$$\frac{1}{a^2} = \int \frac{d^2k}{(2\pi)^2} P(k) \quad (3.12)$$

We will denote the set of all allowed functions by \hat{S} . \hat{S} is the set of all finite linear combinations of the functions $e^{ik \cdot x}$ with $k \in S$. Note that we have $\hat{h}(k) = 0$ if $h \in \hat{S}$ and $k \notin S$.

We now define the partition function as:

$$Z = \int Dh e^H \quad (3.13)$$

Where the measure Dh on \hat{S} is defined as:

$$Dh \equiv \prod_{k \in S} \frac{d\hat{h}(k)}{a} R(k) \quad (3.14)$$

The function $R(k)$ which occurs in the definition of the measure has to be chosen such that the free energy of the exactly soluble Gaussian model is consistent with the renormalization group equation for the free energy. Although the correct choice of $R(k)$ is important for a consistent description of the theory, it turns out that its effect is equivalent to adding a constant term independent of any couplings to the Hamiltonian and hence doesn't influence the dependence of the free energy on the couplings.

3.3 Renormalization

The renormalized Hamiltonian is obtained from (3.6) by using the Wilson-Kogut momentum shell integration technique [29, 37]. We will integrate (3.13) over some of the degrees of freedom, leaving us with an effective Hamiltonian (\tilde{H}) with a lower cut-off. Next a scale transformation will restore the original cut-off and yield the renormalized Hamiltonian (H_R).

We must now specify precisely the degrees of freedom we have to integrate over. Since the renormalized Hamiltonian (H_R) has the same cut-off function $P(k)$ as the original Hamiltonian (H), and since it is obtained from the effective Hamiltonian (\tilde{H}) after a scale transformation, \tilde{H} has to have a cut-off function of the form $P(lk)$. In terms of l the scale transformation becomes $x \rightarrow l^{-1}x$. We thus have to construct a set $S^{(1)}$ of allowed k -values for \tilde{H} , such that $S^{(1)} \subset S$ and $S^{(1)}$ corresponds to the cut-off function $P(lk)$. The complement of $S^{(1)}$ in S , denoted as $S^{(2)}$, contains the degrees of freedom we have to integrate over. We thus have to split the set S of k -values into two disjoint sets $S^{(1)}$ and $S^{(2)}$. This can be done as follows: We decide to put the points $k \in S$ and $-k \in S$ in $S^{(1)}$ with probability $\frac{P(lk)}{P(k)}$. $S^{(2)}$ is defined as $S^{(2)} = S - S^{(1)}$. The cut-off function for $S^{(2)}$ will be denoted as $P^{(2)}$, is thus given by

$$P^{(2)} = P(k) - P(lk) \quad (3.15)$$

We now construct the spaces $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ analogous to \hat{S} : $\hat{S}^{(i)}$ is defined as the set of all finite linear combinations of the functions $e^{ik \cdot x}$ with $k \in S^{(i)}$. We now have

$$\hat{S} = \hat{S}^{(1)} \oplus \hat{S}^{(2)} \quad (3.16)$$

The projection of a $h \in \hat{S}$ on $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ will be denoted by $h^{(1)}$ respectively $h^{(2)}$. The first step in the Wilson-Kogut renormalization scheme is to integrate

over the field $h^{(2)}$. After this integration one obtains an effective Hamiltonian \tilde{H} which depends on $h^{(1)}$. The final step is to restore the original cut-off by a length rescaling:

$$x' = l^{-1}x \quad (3.17)$$

The renormalized field h^R is defined as:

$$h^R(x') = h^{(1)}(x) \quad (3.18)$$

and the renormalized Hamiltonian H_R is defined as:

$$H_R(h^R) = \tilde{H}(h^{(1)}) \quad (3.19)$$

3.4 Cumulant expansion

The integration over the field $h^{(2)}$ is performed after an expansion about the Gaussian model. We rewrite our Hamiltonian (3.6) as

$$H = H_g + X \quad (3.20)$$

where H_g is a Gaussian interaction and X is a perturbation. H_g may be split into a Gaussian interaction for $h^{(1)}$ and $h^{(2)}$, denoted as $H^{(1)}$ respectively $H^{(2)}$

$$\begin{aligned} H_g &= -\frac{i}{2} \int (\nabla h)^2 d^2x = -\frac{i}{2} \sum_{k \in S} k^2 |h(k)|^2 \\ &= -\frac{i}{2} \sum_{k \in S^{(1)}} k^2 |h(k)|^2 - \frac{i}{2} \sum_{k \in S^{(2)}} k^2 |h(k)|^2 \\ &= -\frac{i}{2} \int (\nabla h^{(1)})^2 d^2x - \frac{i}{2} \int (\nabla h^{(2)})^2 d^2x \\ &= H^{(1)} + H^{(2)} \end{aligned} \quad (3.21)$$

Note that for a given Hamiltonian the representation (3.20) is not unique because one may choose to include a Gaussian term in the perturbation X as well. Such a freedom of choice can sometimes be exploited in first order calculations to improve the accuracy of calculations (see [33]).

We define the measure $Dh^{(2)}$ by

$$\int Dh^{(2)} F(h) \equiv \frac{\int_{h \in \hat{S}^{(2)}} Dh F(h)}{\int_{h \in \hat{S}^{(2)}} Dhe^{H^{(2)}}} = \frac{\int \prod_{k \in S^{(2)}} d\hat{h}(k) F(h)}{\int \prod_{k \in S^{(2)}} d\hat{h}(k) \exp(H^{(2)}(h^{(2)}))} \quad (3.22)$$

where $F(h)$ is an arbitrary function of h . The Gaussian average of a function F over the field $h^{(2)}$ can now be written as

$$\langle F(h) \rangle = \int Dh^{(2)} F(h) \exp(H^{(2)}(h^{(2)})) \quad (3.23)$$

We now define the effective Hamiltonian $\tilde{H}(h^{(1)})$ as follows:

$$\exp(\tilde{H}(h^{(1)})) = K \int Dh^{(2)} \exp(H) \quad (3.24)$$

Here K is a constant. To determine H_R one simply has to rescale \tilde{H} (see (3.17), (3.18) and (3.19)). To fix the constant K , one has to express H_R and H in the

same functional form and then require the constant terms to be equal. From (3.20), (3.21), (3.23) and (3.24) it follows

$$\tilde{H} = \ln(K) + H^{(1)} + \ln \langle \exp(X) \rangle \quad (3.25)$$

To second order in X , (3.25) can be written as

$$\tilde{H} = \ln(K) + H^{(1)} + \langle X \rangle + \frac{1}{2} \langle (X - \langle X \rangle)^2 \rangle + \dots \quad (3.26)$$

This expansion is known as the cumulant expansion. For the general form of this expansion, see [14].

3.5 Diagrammatic expansion

It turns out that the terms in the cumulant expansion can be represented as amplitudes of Feynman-diagrams. In these diagrams the correlation function of the field $h^{(2)}$ plays the role of the propagator. In section B.1 we show that it takes the form:

$$G(x) = \frac{1}{jV} \sum_{k \in S^{(2)}} \frac{\exp(\imath k \cdot x)}{k^2} = \frac{1}{j} \int \frac{d^2 k}{(2\pi)^2} P_c^{(2)}(k) \frac{\exp(\imath k \cdot x)}{k^2} \quad (3.27)$$

where $P_c^{(2)}$ is the characteristic function of the set $S^{(2)}$. The amplitudes of Feynman-diagrams we will encounter later can be expressed as integrals of products of propagators. We have to be carefull with replacing $P_c^{(2)}$ by $P^{(2)}$ in such cases. E.g. we have

$$\int d^2 x \{G(x)\}^2 = \frac{1}{j^2} \int \frac{d^2 k}{(2\pi)^2} \frac{P^{(2)}(k)}{k^4} \quad (3.28)$$

It is not difficult to see that $G(0)$ is universal:

$$G(0) = \frac{1}{2\pi j} \ln(l) \quad (3.29)$$

We now consider the case of an infinitesimal cut-off change:

$$l^{-1} = 1 - \epsilon \quad (3.30)$$

(3.17) becomes

$$x' = (1 - \epsilon)x \quad (3.31)$$

We now associate ϵ with an infinitesimal increase in a rescaling parameter t . The renormalization process then generates one parameter families of Hamiltonians $H(t)$. The renormalization group equations can then be written as

$$\frac{dH}{dt} = \text{coefficient of } \epsilon \text{ in } H_R \quad (3.32)$$

The parameter t is related to a length transformation:

$$x(t) = e^{-t} x(0) \quad (3.33)$$

Instead of the Hamiltonian it is often more convenient to write the renormalization group equations in terms of the Hamiltonian density. We shall denote the effective Hamiltonian density corresponding to the effective Hamiltonian \tilde{H} as \tilde{F} :

$$\tilde{H}(h^{(1)}) = \int d^2x \tilde{F}(h^{(1)}, \partial_i h^{(1)}) \quad (3.34)$$

The renormalized Hamiltonian density, denoted as \tilde{F} , can thus be expressed in terms of \tilde{F} by rewriting (3.34) in terms of h^R :

$$\begin{aligned} H_R(h^R) \equiv \tilde{H}(h^{(1)}) &= \int d^2x \tilde{F}(h^{(1)}, \partial_i h^{(1)}) \\ &= \int d^2x' (1+2\epsilon) \tilde{F}(h^R, (1-\epsilon)\partial_i h^R \dots) \end{aligned} \quad (3.35)$$

where in the last line we used the transformation $x' = (1-\epsilon)x$ and $h^R(x') = h^{(1)}(x)$. The renormalized Hamiltonian density (F_R) can thus be expressed as

$$F_R(h^R) = (1+2\epsilon) \tilde{F}(h^R, (1-\epsilon)\partial_i h^R \dots) \quad (3.36)$$

The renormalization group equations can thus be expressed as

$$\frac{dF}{dt} = \text{coefficient of } \epsilon \text{ in } F_R \quad (3.37)$$

We now proceed with the derivation of the Feynman-rules for the cumulant expansion (3.26). It is convenient to derive these rules first for the term $\langle e^X \rangle$. We shall see that $\ln \langle e^X \rangle$ is obtained by summing over connected diagrams only. Let $F(h, \partial_i h, \partial_{ij} h, \dots)$ be the non-Gaussian part of the Hamiltonian density in (3.6). We can then write:

$$\frac{1}{n!} \langle X^n \rangle = \frac{1}{n!} \int \prod_{k=1}^n d^2x_k \left\langle \prod_{k=1}^n F(h(x_k), \partial_i h|_{x_k}, \partial_{ij} h|_{x_k}, \dots) \right\rangle \quad (3.38)$$

We can evaluate (3.38) by writing F , considered as a function of the field h and its derivatives, as a Fourier integral. We will define a Fourier transform of F as follows:

$$\begin{aligned} \hat{F}(\gamma, \gamma_i, \gamma_{ij}, \dots) &= \int \frac{dh}{2\pi} \int \prod_i \frac{d\partial_i h}{2\pi} \int \prod_{ij} \frac{d\partial_{ij} h}{2\pi} \dots \\ &F(h, \partial_i h, \partial_{ij} h, \dots) e^{-i[\gamma h + \gamma_i \partial_i h + \gamma_{ij} \partial_{ij} h \dots]} \end{aligned} \quad (3.39)$$

The integrals in (3.39) are from $-\infty$ to ∞ . F can now be written as

$$\begin{aligned} F(h, \partial_i h, \partial_{ij} h, \dots) &= \int d\gamma \int \prod_i d\gamma_i \int \prod_{ij} d\gamma_{ij} \dots \\ &\hat{F}(\gamma, \gamma_i, \gamma_{ij}, \dots) e^{i[\gamma h + \gamma_i \partial_i h + \gamma_{ij} \partial_{ij} h \dots]} \end{aligned} \quad (3.40)$$

The next step is to substitute the representation (3.40) for the Hamiltonian density in (3.38). To facilitate this, it is convenient to introduce multi-indices. The term in the exponent in (3.40) can be written as

$$\gamma h + \sum_{k=1}^{\infty} \gamma_{i_1, \dots, i_k} \partial_{i_1, \dots, i_k} h \quad (3.41)$$

A tuple of k indices, as in the summation in (3.41), can be treated as a single index. Such an index is called a multi-index. A tuple of k indices will be written as (k) . We can thus rewrite (3.41) as

$$\sum_{k=0}^{\infty} \gamma_{(k)} \partial_{(k)} h \quad (3.42)$$

Note that repeated multi-indices are only summed over while keeping the number of indices contained in the multi-index constant. See section A.3 for all the conventions on multi-indices. Inserting (3.40) in (3.38) gives

$$\frac{1}{n!} \langle X^n \rangle = \frac{1}{n!} \int \left(\prod_{j=1}^n d^2 x_j d \left\{ \gamma^{(j)} \right\} \right) \left(\prod_{j=1}^n \hat{F} \left(\left\{ \gamma^{(j)} \right\} \right) \right) \left\langle e^{i \sum_{j=1}^n \gamma_{(k)}^{(j)} \partial_{(k)} h(x_j)} \right\rangle \quad (3.43)$$

Note that the term in the exponent in (3.43) can be interpreted as the action of a distribution (i.e. a linear functional) on the field h . The action of a distribution T on a function h is denoted as Th . See section A.5 of the appendix for a precise definition of distributions. The problem is thus to evaluate

$$\langle e^{iTh} \rangle \quad (3.44)$$

for a general distribution T . In chapter C of the appendix it is shown that

$$\langle e^{iTh} \rangle = e^{-\frac{1}{2} T_x T_y G(x-y)} e^{iTh^{(1)}} \quad (3.45)$$

Here T denotes a distribution, T_x and T_y act as T on $G(x-y)$ considered as a function of x respectively y (x and y are thus “dummy”-variables).

We now have to expand (3.45) in powers of the propagator, and substitute the result in (3.43). The distribution T in (3.45) is defined as follows: First we define the distribution $T^{(j)}$ as

$$T^{(j)} h = \sum_{k=0}^{\infty} \gamma_{(k)}^{(j)} \partial_{(k)} h(x_j) \quad (3.46)$$

The distribution T is then defined as

$$T = \sum_{j=1}^n T^{(j)} \quad (3.47)$$

The L^{th} order term in the propagator in the integrand of (3.43) becomes

$$\frac{1}{n!} \frac{(-1)^L}{2^L L!} [T_x T_y G(x-y)]^L \left(\prod_{j=1}^n \hat{F} \left(\left\{ \gamma^{(j)} \right\} \right) \right) e^{iTh^{(1)}} \quad (3.48)$$

Each term in the expansion of $[T_x T_y G(x-y)]^L$ can be represented diagrammatically. We first perform a trivial step:

$$[T_x T_y G(x-y)]^L = \prod_{p=1}^L T_x T_y G(x-y) \quad (3.49)$$

Each term in the expansion of the product can be represented diagrammatically as follows. Draw the N points x_j . If we choose from the p^{th} term in the product the term $\gamma_{(m)}^{(r)} \partial_{(m)}^{(x_r)}$ from T_x and the term $\gamma_{(n)}^{(s)} \partial_{(n)}^{(x_s)}$ from T_y , we draw an oriented line from x_r to x_s , we label the line with the value of p , and at the points x_r and x_s we put the labels (m) respectively (n) on the line. We repeat this for all values of p . There is now a one to one correspondence between the set of all possible terms in the expansion of the product and the set of labelled diagrams. The amplitude of a labelled diagram is obtained by inserting the appropriate product of the γ 's and the derivatives of the propagators in (3.48). We see that the integrals over the γ 's result in a factor

$$\frac{1}{i^r} \frac{\partial^r F}{\partial (\partial_{(m_1)} h) \cdots \partial (\partial_{(m_r)} h)} \Big|_{h=h^{(1)}} \quad (3.50)$$

for each vertex where r lines, labelled by $(m_1) \dots (m_r)$, come together. The product of the factors $\frac{1}{i^r}$ at each vertex will precisely cancel the factor $(-1)^L$ in (3.48), because each propagator gives rise to two factors $\frac{1}{i}$ and there are L propagators. We can simplify matters further by omitting all labels, except the multi-indices at both ends of each propagator, in a labelled diagram. The amplitude of such a Feynman diagram is given by the sum of all the corresponding labelled diagrams. It is convenient to define a propagator $G_{(n),(m)}(p-q)$ as

$$G_{(n),(m)}(p-q) \equiv \partial_{(n),x}^{(p)} \partial_{(m),y}^{(q)} G(x-y) \quad (3.51)$$

Since all labelled diagrams corresponding to a nonvanishing Feynman diagram make identical contributions, we simply have to multiply the amplitude of one diagram by the number of ways of labeling a Feynman diagram, to obtain its amplitude (relabeling the vertices will change the amplitude of a diagram, but when integrated over all positions of the vertices, all diagrams obtained from each other by a relabeling of the vertices will, of course, make identical contributions). This amplitude then has to be integrated over all the x_j . We shall denote the number of ways of orienting the propagators, labeling the propagators and the vertices by respectively N_1 , N_2 and N_3 . Since the multi-indices have to be summed over, two labelings of the propagators will not be considered distinct if the only difference is a permutation of the multi-indices. Two labelings of the vertices are considered distinct if it is not possible to transform one labeling into the other by a relabeling of the propagators. We then have

$$N_1 = 2^{L-k} \quad (3.52)$$

where k is the number of lines that have both their ends connected to the same vertex,

$$N_2 = \frac{L!}{\prod_r k_r!} \quad (3.53)$$

where the product is over all ordered pairs of vertices, and k_r denotes the number of propagators connecting the pair r and

$$N_3 = \frac{n!}{S} \quad (3.54)$$

where S is the order of the symmetry group of the Feynman diagram. Using (3.48), (3.50), (3.51), (3.52), (3.53) and (3.54), we see that the Feynman rules for $\langle e^X \rangle$ are as follows:

1. To compute the contribution that is n^{th} order in X and L^{th} order in $\frac{1}{j}$, draw all topological distinct Feynman diagrams with n vertices and L lines.
2. Label both ends of each line by arbitrary multi-indices.
3. For each vertex there is a term:

$$\left. \frac{\partial^r F}{\partial (\partial_{(m_1)} h) \cdots \partial (\partial_{(m_r)} h)} \right|_{h=h^{(1)}} \quad (3.55)$$

where the (m_i) are the multi-indices on the lines at the vertex and the derivative is evaluated at the coordinates of the vertex.

4. Each line labelled with the multi-indices (m) and (n) corresponds to the propagator $G_{(n),(m)}(p-q)$:

$$G_{(n),(m)}(p-q) = \partial_{(n),x}^{(p)} \partial_{(m),y}^{(q)} G(x-y) \quad (3.56)$$

where p and q are the coordinates of the vertices connected by the line.

5. For each line that has both its ends connected to the same point there is a factor $\frac{1}{2}$.
6. For each pair of vertices connected by k lines there is a factor $\frac{1}{k!}$.
7. There is a factor $\frac{1}{S}$, where S is the order of the symmetry group acting on the vertices of the diagram.
8. Integrate over all coordinates of the vertices, and sum over all multi-indices.

We will now show that $\ln \langle e^X \rangle$ is precisely the sum of all connected diagrams. We assume that all connected diagrams are enumerated in some arbitrary order. Let C_i be the amplitude of the i^{th} connected diagram. Using the above Feynman rules, we can write:

$$\langle \exp(X) \rangle = \sum_{\{n_i\}} \prod_{i=1}^{\infty} \frac{C_i^{n_i}}{n_i!} = \prod_{i=1}^{\infty} \sum_{n=0}^{\infty} \frac{C_i^n}{n!} = \exp \left(\sum_{i=1}^{\infty} C_i \right) \quad (3.57)$$

3.6 Evaluation of diagrams

There are two diagrams, see fig. 3.1, contributing to the first order cumulant. Using the Feynman rules derived above it is a simple matter to evaluate the amplitude of these diagrams. In the case of a Hamiltonian density $F(h, \partial_i h \cdots)$ one obtains to first order the effective Hamiltonian density \tilde{F}

$$\tilde{F}(h^{(1)}, \partial_i h^{(1)} \cdots) = F(h^{(1)}, \partial_i h^{(1)} \cdots) + \frac{1}{2} \sum_{l,m} \left[\frac{\partial^2 F}{\partial_{(k)} h \partial_{(l)} h} \right]_{h=h^{(1)}} G_{(l),(m)}(0) \quad (3.58)$$

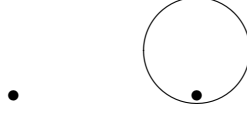


Figure 3.1: The two Feynman diagrams corresponding to the first order cumulant.

where the sum over l and m is from 0 to ∞ . To obtain the renormalization group equation for the Hamiltonian density from this, we have to perform a rescaling $x \rightarrow (1 - \epsilon) x$ (see (3.36)) and use (3.37). These equations yield the first order renormalization group equation for the Hamiltonian density:

$$\frac{dF}{dt} = 2F - \sum_{k=1}^{\infty} k \frac{\partial F}{\partial (\partial_{(k)} h)} \partial_{(k)} h + \frac{1}{2} \sum_{k,l} \frac{\partial^2 F}{\partial (\partial_{(k)} h) \partial (\partial_{(l)} h)} \frac{G_{(k),(l)}}{\epsilon} \quad (3.59)$$

The sum over k and l is again from 0 to ∞ . The quantity $G_{(k),(l)}(0)$ is universal (i.e. independent of the form of the cut-off function $P(k)$) when $k = l = 0$ or $k + l = 2$. It is not difficult to derive the result:

$$G_{(l),(m)}(0) = (-1)^{\frac{l+m}{2}} \frac{1}{2\pi j} \frac{1}{2^{\frac{l+m}{2}} \left(\frac{l+m}{2}\right)!} A_{l+m} C_{(l),(m)} \quad (3.60)$$

Here A_n is zero if n is odd, else we have

$$A_n = \int_0^\infty d|k| |k|^{n-1} (P(|k|) - P(|(1 + \epsilon)k|)) \quad (3.61)$$

In particular we have:

$$\begin{aligned} A_0 &= \epsilon \\ A_2 &= \frac{4\pi}{a^2} \epsilon \end{aligned} \quad (3.62)$$

The tensor $C_{(l),(m)}$ is a contraction operator. For an arbitrary tensor $T_{(l),(m)}$, $T_{(l),(m)} C_{(l),(m)}$ is the sum of all contractions of the indices contained in (l) and (m) . We can write $C_{(l),(m)}$ explicitly as a sum of products of Kronecker delta's:

$$C_{(l),(m)} = \sum_{\pi} \delta'_{\pi(i_1), \pi(i_2)} \cdots \delta_{\pi(i_{l+m-1}), \pi(i_{l+m})} \quad (3.63)$$

The sum is over all nonequivalent permutations of the indices $i_1 \cdots i_{l+m}$. There are thus $\frac{(l+m)!}{2^{\frac{l+m}{2}} \left(\frac{l+m}{2}\right)!}$ terms in the sum.

We now proceed with the evaluation of the higher order cumulants. According to the Feynman rules the n^{th} order contribution to the renormalized Hamiltonian is an n -fold integral over the volume of a product of propagators and functions of the field $h^{(1)}$. We want to replace such an expression by a single integral over the volume, thus obtaining a contribution to the Hamiltonian density. We write the amplitude A of a diagram as

$$A = \int d^2 x_1 \cdots d^2 x_n P(x_1 \cdots x_n) D(h^{(1)}(x_1) \cdots h^{(1)}(x_n)) \quad (3.64)$$

Eigenoperator	Scale dimension
1	0
$\cos(\pi h)$	$\frac{\pi}{4j}$
$a^2 (\nabla h)^2 - \frac{1}{j}$	2
$\left(a^2 (\nabla h)^2 - \frac{1}{j}\right) \cos(\pi h)$	$\frac{\pi}{4j} + 2$
$\cos(2\pi h)$	$\frac{\pi}{j}$

Table 3.1: Some eigenoperators and their scale dimensions relative to the Gaussian Hamiltonian $H_g = -\frac{j}{2} \int d^2x (\nabla h)^2$.

Here $P(x_1 \cdots x_n)$ is the product of propagators and $D(h^{(1)}(x_1) \cdots h^{(1)}(x_n))$ denotes the product of derivatives of Hamiltonian densities. It is now tempting to perform $n - 1$ of the n integrations in (3.64) by Taylor-expanding the field about one of the points $x_1 \cdots x_n$ (it doesn't matter which integrations are performed because different choices are related by a partial integration). The problem with this approach is that it assumes that the field $h^{(1)}$ is analytic. In reality one should expect a Taylor-expansion of the field to converge only in a region the size of a (3.12), because a is the distance between independent degrees of freedom of the field.

A better way to proceed is to use the so-called operator product expansion. Before we explain how this works we will first introduce some new terminology. A local operator is a term in the Hamiltonian that depends only on the field in one point. The Hamiltonian density evaluated at a certain point is an example of a local operator. The first order renormalization group equations for local operators is almost identical to that of the Hamiltonian density. If $O(h(x))$ is a local operator then

$$\frac{dO}{dt} = - \sum_{k=1}^{\infty} k \frac{\partial O}{\partial (\partial_{(k)} h)} \partial_{(k)} h + \frac{1}{2} \sum_{k,l} \frac{\partial^2 O}{\partial (\partial_{(k)} h) \partial (\partial_{(l)} h)} \frac{G_{(k),(l)}}{\epsilon} \quad (3.65)$$

The only difference with (3.59) is that the term $2O$ doesn't appear on the r.h.s. of this equation. Eigenoperators are local operators that renormalize as

$$\frac{dO}{dt} = -\lambda O \quad (3.66)$$

λ is called the scale dimension of the operator O . In table 3.1 we have listed a few eigenoperators with their scale dimensions. By solving (3.66) for all eigenoperators one obtains a complete set of operators. All eigenoperators can be written as a multinomial of derivatives of the field h multiplied by $\cos(n\pi h)$ with n an integer. We shall call an operator even (odd) if n is even (odd). We can now expand any local operator in this set of eigenoperators. A product of operators localized at different points can be considered to be local if the points lie close to each other. This product can then be expanded in eigenoperators localized at one point. It is clear that this expansion, known as the operator product expansion, can be used to replace $D(h^{(1)}(x_1) \cdots h^{(1)}(x_n))$ in (3.64) by a sum of operators localized at the point x_1 . Suppose all eigenoperators are enumerated by an index n . The scale dimension of the n^{th} operator will be

denoted as λ_n . We can thus put

$$D(h(x_1) \cdots h(x_n)) \equiv \sum_k c_k(x_1 \cdots x_n) O_k(h(x_1)) \quad (3.67)$$

Note that we have replaced the field $h^{(1)}$ by h . To apply (3.67) to (3.64) a rescaling must thus be performed first. It is important to note that (3.67) is an identity in the sense that the Hamiltonian to which the l.h.s. is added may be identified with the Hamiltonian to which the r.h.s. is added. Since $D(h(x_1) \cdots h(x_n))$ is a sum of products of eigenoperators located at the points $x_1 \cdots x_n$, all we need to know are the functions $c_{i_1 \cdots i_n, k}(x_1 \cdots x_n)$ (operator product expansion coefficients) in the expansion

$$O_{i_1}(h(x_1)) \cdots O_{i_n}(h(x_n)) \equiv \sum_k c_{i_1 \cdots i_n, k}(x_1 \cdots x_n) O_k(h(x_1)) \quad (3.68)$$

The operator product expansion coefficients can be determined as follows. One demands that the replacement of the product of eigenoperators according to (3.68) commutes with a renormalization. One then obtains an equation relating $c_{i_1 \cdots i_n, k}(x_1 \cdots x_n)$ to $c_{i_1 \cdots i_n, k}(\frac{x_1}{l} \cdots \frac{x_n}{l})$, with l the rescaling factor involved in the renormalization. Now, when one takes $x_1 = x_2 = \cdots = x_n$ the operator product expansion is trivial. The functions $c_{i_1 \cdots i_n, k}(x_1 \cdots x_n)$ can thus be determined by taking the limit $l \rightarrow \infty$. Note that that since the renormalization has to be carried out perturbatively one obtains the operator product expansion coefficients as an expansion in the non-Gaussian couplings. It is thus very straightforward to find the operator product expansion coefficients to zeroth order. Higher order contributions to the operator product expansion coefficients will again involve nontrivial integrations over Feynman-diagrams. These diagrams must again be evaluated using the operator product expansion. E.g. to find the renormalization group equations to second order one has to deal with expressions as in (3.64) with $n = 2$. Since the function D is already of second order one only has to work out the operator product expansion to zeroth order, which is straightforward. To third order one has to calculate the operator product expansion coefficients in (3.68) with $n = 3$ to zeroth order and the operator product expansion coefficients with $n = 2$ to first order. The latter ones involve Feynman-diagrams in which the two operators are connected to one of the other operators in the Hamiltonian. These diagrams can be evaluated by again using the operator product expansion (3.68), but now with $n = 3$ and only to zeroth order. It is clear that repeated use of the operator product expansion allows one to obtain the renormalization group equations to any order.

in the next chapter, we are going to apply the theory to find the phase diagram of the staggered F-model.

Chapter 4

Applications of the renormalization group equations

Once the renormalization group equations are known it is a simple matter to obtain the singular part of the free energy. In this section we shall first derive the renormalization group equation for the free energy and then proceed to show how the singular part of the free energy is obtained from it.

It is convenient to rewrite the Hamiltonian as

$$H = -\frac{j}{2} \int d^2x (\nabla h)^2 + \sum_{n=1}^{\infty} y_n \int \frac{d^2x}{a^2} O_n(h(x)) \quad (4.1)$$

Here the O_n are the eigenoperators defined by equations (3.65) and (3.66). We write the renormalization group equations as

$$\frac{dy_i}{dt} = (2 - \lambda_i) y_i + \sum_{k=2}^{\infty} \sum_{i_1 \dots i_k} \lambda_{i, i_1 \dots i_k} y_{i_1} \dots y_{i_k} \quad (4.2)$$

Note that the λ_i are the scale dimensions of the operators. Also note that the Gaussian coupling is kept constant under renormalization. This is possible because $a^2 (\nabla h)^2 - \frac{1}{j}$ is an eigenoperator. To find out how the singularity in the free energy is related to the λ 's in this equation, we must first find the relation between the free energy of the original and the renormalized system. Note that the renormalized Hamiltonian satisfies the relation

$$\frac{\exp(H_R)}{Z_R} = \int_{h \in \hat{S}^{(2)}} \frac{\exp(H)}{Z} Dh \quad (4.3)$$

where Z_R and Z are the partition functions for the renormalized respectively the original system. Combining (3.22) with (3.24) and (4.3) we get:

$$\frac{Z_R}{Z} = \frac{K}{Z_g} \quad (4.4)$$

Here Z_g is the partition function of the Gaussian model:

$$Z_g = \int_{h \in \hat{S}^{(2)}} \exp\left(H^{(2)}\right) Dh \quad (4.5)$$

Let U be the constant contribution to H_R from $\ln\langle\exp(X)\rangle$. Since the total constant contribution to H_R is zero, it follows from (3.25) that $\ln(K) = -U$. (4.4) can then be written

$$F = (1 - 2\epsilon) F_R + \frac{U}{V} + \frac{1}{V} \ln(Z_g) \quad (4.6)$$

where F and F_R are the free energy densities times $-\beta$ for the original respectively renormalized system.

$$\frac{dF}{dt} - 2F + c + c' = 0 \quad (4.7)$$

Here c is the coefficient of ϵ in $\frac{U}{V}$ and c' is the coefficient of ϵ in $\frac{1}{V} \ln(Z_g)$. Since c' only depends on j , which is kept constant under renormalization, the effect of this term is to shift the free energy by a constant amount. We are thus allowed to ignore this term.

4.1 The case of the staggered F-model

When $\beta s = 0$ and $\beta\epsilon < \ln(2)$, it is known that the F-model renormalizes to the Gaussian model [12, 16, 17]. For the latter we have

$$H = \frac{-j}{2} \int d^2x (\nabla h)^2 \quad (4.8)$$

The Gaussian coupling j is a known function of the temperature of the F-model:

$$j = \frac{1}{2} \arccos\left(1 - \frac{1}{2} e^{2\beta\epsilon}\right) \quad (4.9)$$

(4.9) is valid when $\beta\epsilon < \ln(2)$ and is obtained as follows: The long range part of the height-height correlation function $R(r) \equiv \langle (h(r) - h(0))^2 \rangle$ of both models show a logarithmic behaviour, and is thus invariant under horizontal scaling. This means that the amplitude of the height-height correlation function is invariant under a renormalization. For the F-model one finds [40]

$$R(r) \sim \frac{2}{\pi \arccos\left(1 - \frac{1}{2} \exp(2\beta\epsilon)\right)} \ln(r) \quad (4.10)$$

In case of the Gaussian model one finds (see section B.2)

$$R(r) \sim \frac{1}{\pi j} \ln(r) \quad (4.11)$$

Equating the amplitudes of both correlation functions then leads to the identification (4.9).

We expect that when $\beta s \approx 0$, we may replace (3.4) by a Hamiltonian of the form (4.1). Then because of (3.5) the y_n multiplying even (odd) operators will be even (odd) functions of βs . We now assume that the y_n in (4.1) are analytic in some neighborhood of $\beta s = 0$. This implies that the y_n corresponding to odd operators are $O(\beta s)$. Of all operators, the operator $O_1(h) = \cos(\pi h)$ has the lowest scale dimension:

$$\lambda_1 = \frac{\pi}{4j} \quad (4.12)$$

Between $j = \frac{\pi}{8}$ and $j = \frac{\pi}{2}$ this is the only relevant operator (i.e. an initially infinitesimal y_1 increases exponentially under renormalization). Below $j = \frac{\pi}{8}$ there are no relevant operators and above $j = \frac{\pi}{2}$ the operator $\cos(2\pi h)$ also becomes relevant. Because the coupling y_1 becomes proportional to βs in the limit $\beta s \rightarrow 0$, we expect the staggered F-model with an infinitesimal staggered field to be in a different phase than at zero staggered field for those values for $\beta\epsilon$ that correspond to a value for j between $j = \frac{\pi}{8}$ and $j = \frac{\pi}{2}$. According to (4.9) this is for $\beta\epsilon$ in the interval $\frac{1}{2} \ln(2 - \sqrt{2}) < \beta\epsilon < \ln(2)$. Note that at the lower boundary $\beta\epsilon$ is negative: $\frac{1}{2} \ln(2 - \sqrt{2}) \approx -0.2674$. At zero staggered field the logarithmic behaviour of the height-height correlation function indicates that the surface is in a rough phase. If the staggered field is turned on the model no longer renormalizes to a Gaussian model. If the staggered field is chosen small enough we expect that under a renormalization the model will renormalize first to a model of the form

$$H = \int d^2x \left[\frac{-j}{2} (\nabla h)^2 + \frac{y_1}{a^2} \cos(\pi h) \right] \quad (4.13)$$

with a small value for y_1 , but as we renormalize further the coupling y_1 will increase. Since the effect of the operator $\cos(\pi h)$ in the Hamiltonian is to favour even values of h , we expect to be in a smooth phase. Below $\beta\epsilon = \frac{1}{2} \ln(2 - \sqrt{2})$ we still expect that the model will renormalize to (3.4) but as we renormalize further y_1 will renormalize to zero. We are then left with a purely Gaussian Hamiltonian which describes a rough surface.

Above $\beta\epsilon = \ln(2)$ the operator $\cos(2\pi h)$ becomes relevant. Since this is an even operator its coupling is nonzero at zero staggered field. This causes the model to no longer renormalize to the Gaussian model (as a consequence (4.9) is not valid in this region). If $\beta\epsilon > \ln(2)$ the surface is thus in a smooth phase even if $\beta s = 0$. To complete the phase diagram we must find the behaviour of the model for finite values of the staggered field below $\beta\epsilon = \frac{1}{2} \ln(2 - \sqrt{2})$. Before we do that we shall first calculate the singular part of the free energy above $\beta\epsilon = \frac{1}{2} \ln(2 - \sqrt{2})$ at $\beta s = 0$.

4.2 Singular part of the free energy of the staggered F-model

We shall assume that the staggered F-model can be mapped to a model of the form (4.1) such that the couplings y_n are analytical as a function of βs in a neighborhood of $\beta s = 0$. If the free energy F of the model (4.1) is written as

$$F = F_s + F_r \quad (4.14)$$

with F_s the singular part of the free energy and F_r the regular part of the free energy, F_s will satisfy the homogeneous part of (4.7) and F_r will be a full solution of (4.7). Exceptions to this rule may arise when a critical exponent associated with the singular behaviour of the free energy is an even integer as we shall see later. Ignoring these exceptions for the moment, we see that F_s satisfies the equation:

$$F_s(y_1(t), y_2(t) \dots) = e^{2t} F_s(y_1(0), y_2(0) \dots) \quad (4.15)$$

In order to see how irrelevant operators modify the singular behaviour, it is enough to keep just one irrelevant coupling. The generalization to more irrelevant couplings is trivial. Suppose that for $\beta s \approx 0$ the staggered F-model model is mapped to a model (4.1) with $y_1(0)$ the coupling of the relevant operator $\cos(\pi h)$ and $y_2(0)$ the coupling of an irrelevant even operator. The mapping to the model (4.1) can then be written

$$\begin{aligned} y_1(0) &= R_1 \beta s + O((\beta s)^3) \\ y_n(0) &= R_n + O((\beta s)^2) \end{aligned} \quad (4.16)$$

The renormalization group equations (4.2) can be rewritten as

$$\frac{dy_i}{dt} = a_i y_i \quad (4.17)$$

With $a_1 = 2 - \frac{\pi}{4j} > 0$ and $a_n < 0$. Higher order terms in the renormalization group equations have been ignored. From (4.15) and (4.17) it then follows that

$$F_s(y_1(0), y_2(0)) = e^{-2t} F_s(y_1(0) e^{a_1 t}, y_2(0) e^{a_2 t}) \quad (4.18)$$

Now choose t such that

$$y_1(0) e^{a_1 t} = c \quad (4.19)$$

with c a constant $\neq 0$. we can then rewrite (4.18) as

$$F_s(y_1(0), y_2(0)) = \left(\frac{y_1(0)}{c} \right)^{\frac{2}{a_1}} F_s \left(c, y_2(0) \left(\frac{y_1}{c} \right)^{\frac{-a_2}{a_1}} \right) \quad (4.20)$$

Since we expect F_s to be analytical as a function of $y_2(0)$ as long as $y_1(0) \neq 0$, we can expand:

$$F_s \left(c, y_2(0) \left(\frac{y_1}{c} \right)^{\frac{-a_2}{a_1}} \right) = A + B y_2(0) \left(\frac{y_1}{c} \right)^{\frac{-a_2}{a_1}} + \dots \quad (4.21)$$

Inserting this into (4.20) and using (4.16) yields for the leading singularity in the free energy ($F_1(\beta s)$):

$$F_1(\beta s) \sim |\beta s|^{\frac{2}{a_1}} \quad (4.22)$$

while the irrelevant operator contributes a singularity ($F_2(\beta s)$) of the form

$$F_2(\beta s) \sim F_1(\beta s) |\beta s|^{\frac{-a_2}{a_1}} \quad (4.23)$$

Note that $a_1 = 2 - \frac{\pi}{4j}$, and j is given by (4.9). As $\beta\epsilon$ approaches $\frac{1}{2} \ln(2 - \sqrt{2})$ from above a_1 tends to zero, and the singularity in the free energy becomes weaker and weaker. What happens at $\beta\epsilon = \frac{1}{2} \ln(2 - \sqrt{2})$ and below is the subject of section 4.3. The above argument can easily be generalised to take account of the presence of more irrelevant operators and higher order terms in the identification (4.16), the renormalization group equations (4.17) and the expansion (4.21). By applying a general result [36] to this case, we find that the free energy contains singularities of the form

$$F_s(\beta s) \sim |\beta s|^{n_0 + \frac{2}{a_1} - \sum_{k=2}^{\infty} \frac{n_k a_k}{a_1}} \quad (4.24)$$

where the n_i are positive integers. If the exponent becomes an even integer we have to multiply the r.h.s. of (4.24) with $\ln |\beta s|$. We can demonstrate this in the case of the leading singularity as follows: According to (4.7) the renormalization group equation for the free energy is given by

$$\frac{dF}{dt} - 2F = -c(y_1) \quad (4.25)$$

c is an even analytical function of y_1 , because $\cos(\pi h)$ is an odd operator while the constant operator is even. We are now assuming that $\frac{2}{a_1} = 2n$ where n is an integer. (4.17) gives

$$y_1(t) = y_1(0) e^{a_1 t} \quad (4.26)$$

If $-c$ contains a term $K y_1^{2n}$, (4.25) can be rewritten as

$$\frac{dF}{dt} - 2F = K (y_1(0))^{2n} e^{2t} \quad (4.27)$$

From this equation it follows that

$$F(y(t)) = K (y_1(0))^{2n} t e^{2t} + F(y(0)) \quad (4.28)$$

Using (4.16) and (4.26) it then follows that

$$F_s \sim \beta s^{2n} \ln |\beta s| \quad (4.29)$$

It is interesting to see what happens if we let $\frac{2}{a_1}$ approach the value $2n$. If we put $\frac{2}{a_1} = 2n - \epsilon$ for small ϵ , we can rewrite (4.27) as:

$$\frac{dF}{dt} - 2F = K (y_1(0))^{2n} e^{(2+\epsilon a_1)t} \quad (4.30)$$

expanding the r.h.s. of this equation in powers of ϵ gives

$$\frac{dF}{dt} - 2F = K (y_1(0))^{2n} e^{2t} [1 + \epsilon a_1 t + \dots] \quad (4.31)$$

And the singularity in the free energy can be written as

$$F_s(\beta s) \sim \beta s^{2n} \left[\ln |\beta s| + \frac{\epsilon}{2} \ln^2 |\beta s| + \dots \right] \quad (4.32)$$

4.3 The case $\beta\epsilon \leq \frac{1}{2} \ln(2 - \sqrt{2})$

To complete the phase diagram we must obtain the behaviour of the model at finite values for the staggered field. This requires us to study the effects of higher order terms in the renormalization group equations. To find the most important higher order terms we look for terms that are second order in y_1 . These terms are involved in the generation of operators. The most important of these terms is the one involved in the generation of the most relevant operator. We also look for the lowest order term in the generation of y_1 arising from interactions with operators which have as low an scale dimension as possible. To second order in y_1 only even operators are generated and the most relevant of these is the operator $O_2(h) = a^2 (\nabla h)^2 - \frac{1}{j}$. It is also this operator which through interaction with the operator O_1 contributes to the generation of O_1 , which is also a second order effect. Since O_2 is Gaussian we can calculate this effect simply by perturbing the Gaussian interaction j . Denoting the coupling of O_1 by y and the coupling of O_2 by $-\frac{j'}{2}$, we can write

$$\begin{aligned} \frac{dj'}{dt} &= A(j + j') y^2 \\ \frac{dy}{dt} &= \left(2 - \frac{\pi}{4(j+j')}\right) y \end{aligned} \quad (4.33)$$

Although the function $A(j)$ can be calculated using the methods developed in the previous chapter, for our purpose we can afford to leave this function undetermined. At $j = \frac{\pi}{8}$, which corresponds to $\beta\epsilon \leq \frac{1}{2} \ln(2 - \sqrt{2})$, the operator O_1 is marginal (i.e. right on the boundary between relevant and irrelevant). To investigate the phase diagram around this point, we put $j = \frac{\pi}{8}$ in (4.33) and expand in powers of j' . To leading order we find

$$\begin{aligned} \frac{dj'}{dt} &= Ay^2 \\ \frac{dy}{dt} &= \frac{16}{\pi} j' y \end{aligned} \quad (4.34)$$

where $A \equiv A(\frac{\pi}{8})$. Note that these renormalization group equations are similar to those for the XY model (see [18, 19]). To be able to construct the phase diagram, we must know how to relate j' and y to the model parameters $\beta\epsilon$ and βs of the staggered F-model in a nonzero staggered field. According to (4.16) we can put

$$\begin{aligned} y(0) &= R(\beta\epsilon) \beta s + O((\beta s)^3) \\ j'(0) &= R'(\beta\epsilon) + O((\beta s)^2) \end{aligned} \quad (4.35)$$

where we have used the fact that O_2 is an even operator. The function $R'(\beta\epsilon)$ can be calculated by using the fact that at zero staggered field the model renormalizes to a Gaussian model with coupling j given by (4.9). We thus find that

$$R'(\beta\epsilon) = \frac{1}{2} \arccos\left(1 - \frac{1}{2} e^{2\beta\epsilon}\right) - \frac{\pi}{8} \quad (4.36)$$

We now put $\beta\epsilon = \frac{1}{2} \ln(2 - \sqrt{2}) - u$ in (4.16) and expand to leading order. We find

$$\begin{aligned} y(0) &= R\beta s \\ j'(0) &= -(\sqrt{2} - 1) u \end{aligned} \quad (4.37)$$

where $R \equiv R(\beta\epsilon = \frac{1}{2} \ln(2 - \sqrt{2}))$. According to (4.34) it follows that $K(t)$, defined as

$$K(t) = y(t)^2 - \frac{16}{\pi A} j'(t)^2 \quad (4.38)$$

is a conserved quantity under renormalization. Above $j' = 0$ all flow lines, irrespective of the value of K , renormalize to infinity. Below $j' = 0$ the situation is different. Flow lines with negative K end up on the Gaussian line, while flow lines with positive K renormalize toward infinity. The flow lines with $K = 0$ thus mark the boundary between the rough phase and the smooth phase below $j' = 0$. Using (4.37) and (4.38), we see that the lines

$$\beta s = \pm \frac{4}{R\sqrt{\pi A}} (\sqrt{2} - 1) u \quad (4.39)$$

with $u \geq 0$ are the critical lines of the staggered F-model. Points chosen between these lines renormalize toward the Gaussian line, points outside this region will not.

We now proceed with a derivation the singular part of the free energy. As the critical line is approached from the smooth side, we expect singular behaviour of the free energy (note that points on the critical lines itself renormalize to the point $j' = 0$ on the Gaussian line, there is thus no singularity when the critical line is approached from the rough side). Since all points on the critical lines of the staggered F-model flow toward the same point on the Gaussian line, critical behaviour is the same all along the critical lines. We can thus content ourselves with a calculation of the singular behaviour of the free energy at $\beta\epsilon = \frac{1}{2} \ln(2 - \sqrt{2})$ as we let βs approach zero. In this case we are again in the area where the identification (4.37) and the renormalization group equations (4.34) are valid. The initial values are thus $j'(0) = 0$ and $y(0) = R\beta s$. According to (4.38) we find that $K = R^2(\beta s)^2$ for the streamline that passes through this point. Eliminating $y(t)$ in favour of $j'(t)$ and using (4.34) gives us the equation

$$\frac{dj'}{dt} = AK + \frac{16}{\pi} j'^2 \quad (4.40)$$

This differential equation is easily integrated:

$$t = \frac{\pi}{4\sqrt{A\pi K}} \arctan\left(\frac{4}{\sqrt{A\pi K}} j'(t)\right) \quad (4.41)$$

Using (4.39) and the fact that $K = R^2(\beta s)^2$, we can rewrite (4.41) as

$$t = \frac{\pi(1 + \sqrt{2}) \tan(\theta)}{16|\beta s|} \arctan\left(\frac{(1 + \sqrt{2}) \tan(\theta)}{|\beta s|} j'(t)\right) \quad (4.42)$$

where θ is the angle at which the critical line intersects the line $\beta s = 0$. Applying (4.18) to our case yields the leading singularity in the free energy F_s :

$$F_s(\beta s) \sim e^{-\frac{\pi^2(1+\sqrt{2}) \tan(\theta)}{16|\beta s|}} \quad (4.43)$$

We can see that the singularity is of infinite order. This is characteristic of the Kosterlitz-Thouless transition. Numerical studies using transfer matrix techniques have yielded similar results on the phase diagram of the staggered F-model [24].

Note that all results have been obtained by using the information present in the behaviour of the height-height correlation function of the F-model. To obtain more results we clearly need more information. In the next chapter we shall discuss a simple method that allows one to expand the free energy about the line $\beta\epsilon = \frac{1}{2} \ln(2)$. This expansion can be used to generate more information about the mapping of the staggered F-model to Gaussian models.

Chapter 5

Expansions about free fermion models

In this final chapter we will first present Baxters solution of the staggered F-model on the free fermion line (i.e. the line $\beta\epsilon = \frac{1}{2} \ln(2)$). Then we proceed by expanding the free energy of the staggered F-model about the free fermion line. We shall obtain an explicit expression for the free energy to first order. By comparing the singular behaviour of this expression to that obtained from renormalization group arguments, we are able to verify the known behaviour of the Gaussian coupling to first order about $\beta\epsilon = \frac{1}{2} \ln(2)$. To simplify the computations of the higher order terms we derive a linked cluster method.

5.1 Definition of free fermion models

Baxter has solved the staggered F-model at the temperature $\beta\epsilon = \frac{1}{2} \ln(2)$ [3]. Later it was found that this solution could be generalized to other models if a certain condition concerning the vertex weights is met. This condition is called the free fermion condition because for eight-vertex models satisfying this condition the problem leads to a problem of noninteracting fermions in the S-matrix formulation. Let w_i be the vertex weight for a vertex of type i (see fig. 5.1), then the free fermion condition for six-vertex models is:

$$w_1w_2 + w_3w_4 - w_5w_6 = 0 \tag{5.1}$$

The weights w_i may be chosen inhomogeneous. We now proceed by presenting a simplified version of Baxter's solution of the staggered F-model.

5.2 Baxter's solution of the staggered F-model

Divide the lattice into two sublattices A and B. Choose the vertex energies as indicated in fig. 5.1. Consider the ground state in which all A vertices are vertices of type 6, and all B-vertices are of type 5. Any state can now be represented by drawing lines on the lattice where the arrows point oppositely to the ground state configuration. In terms of these lines the six vertices are represented by vertices with either no lines, two lines at right angles, or four

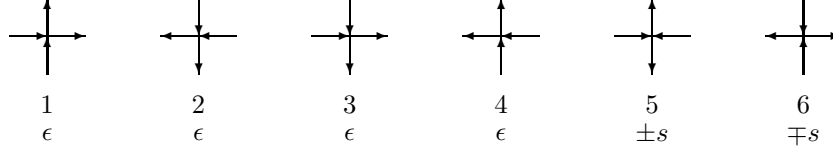


Figure 5.1: The six vertices and their energies. The upper and lower signs correspond to sublattice A respectively B.

lines. The energies of these vertices are respectively $-s$, ϵ and s . The next step is to replace the original lattice by a decorated lattice by replacing each original vertex by a “city” of four internally connected points (see fig. 5.2). The lines on the original lattice are regarded as dimers on the external bonds of the decorated lattice. For any configuration on the original lattice, it is possible to place dimers on the internal bonds of the decorated lattice, so that the lattice becomes completely covered. Now associate to each dimer a weight as indicated in fig. 5.2. We now have to choose these weights such that a close-packed dimer problem formulated on the decorated lattice is equivalent to our original problem. It is a simple matter to see that for this to be the case, we can put

$$C = D = E = F = e^{-\frac{1}{2}\beta s} \quad (5.2)$$

$$u = \frac{1}{2}\sqrt{2}e^{\frac{1}{2}\beta s} \quad (5.3)$$

$$\beta\epsilon = \frac{1}{2}\ln(2) \quad (5.4)$$

Note that (5.4) is indeed consistent with the free fermion condition (5.1).

5.3 The Pfaffian method

To solve the close-packed dimer problem, we use the Pfaffian method [10,13,26]. This method is applicable whenever the lattice is planar, and works by expressing the partition function of the problem as the square root of the determinant of an antisymmetric matrix (a Pfaffian).

A contribution to Z^2 can be written as the product of two dimer coverings C and C' . If C connects a point i with a point j , we write

$$C(i) = j \quad (5.5)$$

It is clear that this defines a bijective map on the lattice. C and C' divide the lattice into disjoint loops and pairs of neighboring points (bonds) as follows: If

$$\begin{aligned} C(i_1) &= i_2 \\ C'(i_2) &= i_3 \\ C(i_3) &= i_4 \\ C'(i_4) &= i_5 \\ C(i_5) &= i_6 \\ &\vdots \\ C(i_{n-1}) &= i_n \\ C'(i_n) &= i_1 \end{aligned} \quad (5.6)$$

then the points $i_1 \dots i_n$ form a loop. If $n = 2$, we don't get a loop but instead a single bond. Note that n is always even (even on lattice types on which loops containing an odd number of points exist, the loops generated by C and C' always contain an even number of points). Since for each loop one has two choices to define the actions of C and C' within the loop, a given partition of the lattice in loops and bonds is consistent with many different configurations C and C' . Z^2 can thus be calculated by summing over all partitions of the lattice in loops and bonds. The contribution a partition makes is given by the appropriate product of the weights of dimers, multiplied by a factor 2^L , where L is the number of loops in the partition. If we orient each loop, and sum over all oriented loops, the factor two for each loop can be omitted. Now a partition of the lattice in oriented loops and bonds defines a permutation of the lattice points. For arbitrary points i and j on the lattice, we define $W_{i,j}$ as

$$\begin{aligned} W_{i,j} &= 0 \text{ if } i \text{ and } j \text{ are not connected,} \\ W_{i,j} &= \text{weight of the dimer connecting } i \text{ and } j \end{aligned} \quad (5.7)$$

In terms of the matrix W , we can write:

$$Z^2 = \sum_{\pi} \prod_j W_{j, \pi(j)} \quad (5.8)$$

where the sum is over all permutations that contain only cycles of even lengths (this restriction is denoted by the prime) and the product is over all lattice points. Note that the restriction on the summation is only necessary for lattices where loops of odd lengths exist. We now want to rewrite the r.h.s. of (5.8) as the determinant of a matrix. This is possible if the lattice is planar, and works as follows: One tries to factorize the missing sign of the permutation π ($s(\pi)$) in the sum in (5.8), so that we have

$$s(\pi) = \prod_j s_{j, \pi(j)} \quad (5.9)$$

with the $s_{i,j}$ depending only on i and j , and $s_{i,j} = \pm 1$ (we only need to define the $s_{i,j}$ when i and j are connected). We shall see that a proper choice of the $s_{i,j}$ allows one to lift the constraint in the summation in (5.8). Anticipating this result we can write:

$$Z^2 = \det R \quad (5.10)$$

where

$$R_{i,j} = s_{i,j} W_{i,j} \quad (5.11)$$

The $s_{i,j}$ have to be chosen such that (5.9) is valid for all permutations making a nonzero contribution to (5.8). The cycles of such a permutation are precisely the oriented loops of even length and bonds, and they all have a sign of -1 . We thus try to define the $s_{i,j}$ such that for a closed loop of even length or a bond consisting of the points $i_1 \dots i_n$ we have

$$\prod_{k=1}^n s_{i_k, i_{k+1}} = -1 \quad (5.12)$$

where $i_{n+1} \equiv i_1$. The case $n = 2$ yields

$$s_{i,j} = -s_{j,i} \quad (5.13)$$

so that R is antisymmetric. We can now see that permutations containing cycles of odd lengths make no net contribution to $\det R$ because reversing such a cycle changes the sign of the contribution. A permutation that contains a cycle with an odd number of points in its interior also makes no net contribution, because, the lattice being planar, these points are permuted amongst themselves, so that the permutation contains at least one cycle of odd length. We thus have to satisfy (5.12) only for loops with an even number of points in its interior. This is fortunate, because it is impossible to choose the $s_{i,j}$ such that (5.12) is satisfied for all loops of even lengths. The $s_{i,j}$ can, however, be chosen to satisfy the condition:

$$\prod_{k=1}^n s_{i_k, i_{k+1}} = (-1)^{r+1} \quad (5.14)$$

where r is the number of points inside the loop. It is clear that if the $s_{i,j}$ satisfy (5.14) for all loops we indeed have $Z^2 = \det R$. We now specialize to the case of the staggered F-model. In this case we are dealing with the lattice shown in fig. 5.2. Choosing the $s_{i,j}$ amounts to giving each bond an orientation so that $s_{i,j}$ is positive if i points to j . The arrows drawn on the bonds in fig. 5.2 represent such an orientation. We will now prove that this choice of the orientations satisfies the condition (5.14). The proof proceeds by induction, and depends on the fact that loops sharing part of their boundaries may be combined to produce larger loops. Note that a loop can be broken down into smaller loops if and only if the loop has bonds in its interior. On the lattice (5.2), there are two types of loops that cannot be broken down. These are the loops formed by four internal bonds of a city, and loops connecting four cities formed by four external bonds and four internal bonds. For these loops it is easily verified that (5.14) is true. Since any loop can be broken down into loops of the above type, we have to prove that if (5.14) holds for two arbitrary loops sharing part of their boundaries, it also holds for the combined loop. To see this, suppose that there are two loops (L_1 and L_2) with respectively r_1 and r_2 interior points, with a continuous common boundary consisting of q points. The combined loop (L_3) will then have $r_3 = r_1 + r_2 + q - 2$ interior points. The product in (5.14) for a loop L_i will be denoted as $s(L_i)$. We then have

$$s(L_1) s(L_2) = s(L_3) (-1)^{q-1} \quad (5.15)$$

because if we traverse L_1 and L_2 in the same direction, we traverse all the bonds of L_3 , while the $q - 1$ bonds on the common boundary are all traversed from both directions. Assuming (5.14) holds for L_1 and L_2 , it follows from (5.15):

$$s(L_3) = (-1)^{r_1+r_2+q-1} = (-1)^{r_3+1} \quad (5.16)$$

5.4 Calculation of the free energy

We have seen that solving the staggered F-model at $\beta\epsilon = \frac{1}{2} \ln(2)$ reduces to the evaluation of the determinant of the matrix R . To set up a perturbation theory

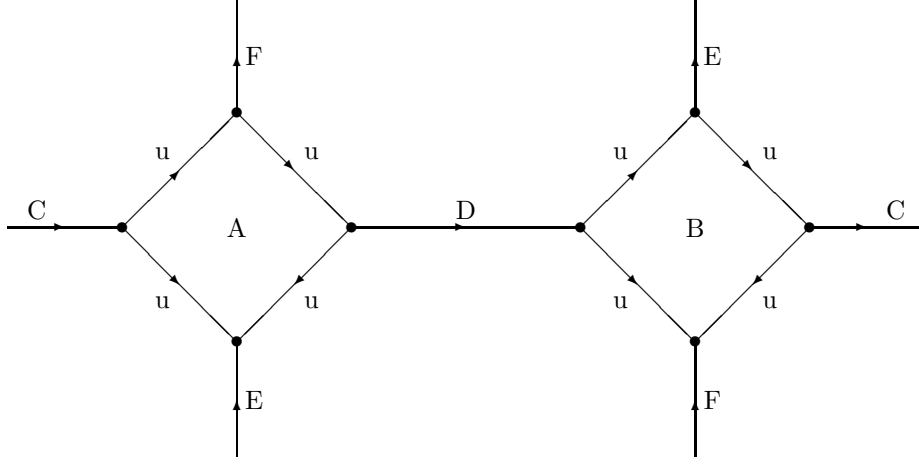


Figure 5.2: The “cities” on the decorated lattice. A and B refer to the two sublattices. The meaning of the orientations on the bonds is explained in the text.

about $\beta\epsilon = \frac{1}{2} \ln(2)$, we also need the inverse of R . Both the determinant and the inverse of R are easily calculated using the following procedure: At each vertex i on the decorated lattice, we associate a variables x_i and x'_i . For x'_i a given set of variables, we attempt to solve the equation:

$$R_{i,j}x_j = x'_i \quad (5.17)$$

We can rewrite this as follows: Introduce coordinates (n, m) on the original lattice, so that increasing n (m) corresponds to moving to the right (upward). To each city on the decorated lattice we assign the coordinates of the corresponding vertex of the original lattice. The variables x_i and x'_i are now given by placing variables $a_{n,m}$, $b_{n,m}$, $c_{n,m}$, $d_{n,m}$ respectively $a'_{n,m}$, $b'_{n,m}$, $c'_{n,m}$, $d'_{n,m}$ for every n and m on the four points of the city with coordinates (n, m) as indicated in fig. 5.3. (5.17) thus becomes

$$\begin{aligned} ua_{n,m} + ub_{n,m} - Cd_{n-1,m} &= c'_{n,m} \\ -ua_{n,m} + ub_{n,m} + Cc_{n+1,m} &= d'_{n,m} \\ Cb_{n,m+1} - uc_{n,m} + ud_{n,m} &= a'_{n,m} \\ -Ca_{n,m-1} - uc_{n,m} - ud_{n,m} &= b'_{n,m} \end{aligned} \quad (5.18)$$

We now perform a Fourier transformation on the variables:

$$a_{p,q} = \sum_{n,m} a_{n,m} e^{-2\pi i(\frac{np}{N} + \frac{mq}{M})} \quad (5.19)$$

The Fourier transform of the other variables is defined similarly. In terms of the Fourier transformed variables (5.18) reads:

$$\begin{aligned} ua_{p,q} + ub_{p,q} - C\beta^{-p}d_{p,q} &= c'_{p,q} \\ -ua_{p,q} + ub_{p,q} + C\beta^p c_{p,q} &= d'_{p,q} \\ C\alpha^q b_{p,q} - uc_{p,q} + ud_{p,q} &= a'_{p,q} \\ -C\alpha^{-q}a_{p,q} - uc_{p,q} - ud_{p,q} &= b'_{p,q} \end{aligned} \quad (5.20)$$

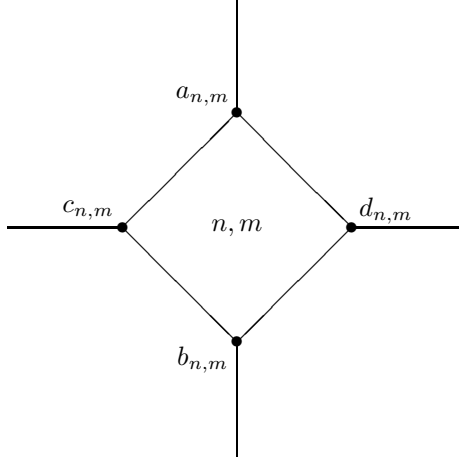


Figure 5.3: The variables $a_{n,m}$, $b_{n,m}$, $c_{n,m}$ and $d_{n,m}$, associated with a city on the decorated lattice.

Here $\alpha = e^{\frac{2\pi i}{M}}$ and $\beta = e^{\frac{2\pi i}{N}}$. The determinant $\Delta_{p,q}$ of (5.20) is given by:

$$\Delta_{p,q} = 2 \cosh(2\beta s) + 2 \cos\left(\frac{2\pi p}{N}\right) \cos\left(\frac{2\pi q}{M}\right) \quad (5.21)$$

And the reduced free energy per vertex (i.e. the free energy times $-\beta$, denoted as F) for an infinite by infinite lattice follows:

$$\begin{aligned} F &= \lim_{N,M \rightarrow \infty} \frac{1}{2NM} \ln \det R = \lim_{N,M \rightarrow \infty} \frac{1}{2NM} \sum_{p,q} \ln \Delta_{p,q} \\ &= \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln [2 \cosh(2\beta s) + 2 \cos(\theta_1) \cos(\theta_2)] d\theta_1 d\theta_2 \end{aligned} \quad (5.22)$$

5.5 Singular part of the free energy

In this section we calculate the singularity in the free energy of the staggered F-model at $\beta s = 0$ on the free fermion line. We shall use the following method: We expand the logarithm in (5.22), thereby obtaining a series of the form

$$F(\beta s) = \sum_{n=1}^{\infty} \frac{1}{\cosh^{2n}(2\beta s)} \left[\frac{A_1}{n^2} + \frac{A_2}{n^3} + \dots \right] \quad (5.23)$$

Summing the series inside the brackets term by term we obtain

$$F(\beta s) = \sum_{k=2}^{\infty} F_k(\beta s) \quad (5.24)$$

with

$$F_k(\beta s) = \sum_{n=1}^{\infty} \frac{1}{n^k \cosh^{2n}(2\beta s)} \quad (5.25)$$

As can be seen by differentiating F_k repeatedly, the singularity in F_k becomes weaker as k increases.

Expanding the logarithm in (5.22) yields

$$\begin{aligned} F(\beta s) &= -\frac{1}{8\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\theta_1 d\theta_2 \sum_{n=1}^{\infty} \frac{\cos^n(\theta_1) \cos^n(\theta_2)}{n \cosh^n(2\beta s)} \\ &= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2n \cosh^{2n}(2\beta s)} \left[\frac{(2n)!}{4^n n!^2} \right]^2 \end{aligned} \quad (5.26)$$

Using

$$n! = n^n e^{-n} \sqrt{2\pi n} \exp \left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \frac{1}{n^{2k-1}} \right) \quad (5.27)$$

where the B_r are the Bernoulli numbers, we find

$$F(\beta s) = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 \cosh^{2n}(2\beta s)} \left[1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{1}{128n^3} - \frac{5}{2048n^4} + \dots \right] \quad (5.28)$$

We can find the singular part of the function $\sum_{n=1}^{\infty} \frac{1}{n^p \cosh^{2n}(2\beta s)}$ as follows: Put $t = \ln(\cosh^2(2\beta s))$. We then have to find the singular part of the function $U_p(t)$ with

$$U_p(t) = \sum_{n=1}^{\infty} \frac{e^{-nt}}{n^p} \quad (5.29)$$

as $t \rightarrow 0$ for $p \geq 2$. From (5.29) it follows that

$$\frac{dU_{p+1}}{dt} = -U_p \quad (5.30)$$

We denote the singular part of U_p by \tilde{U}_p . It then follows from (5.30) that

$$\frac{d\tilde{U}_{p+1}}{dt} = -\tilde{U}_p \quad (5.31)$$

For $p = 1$ the sum in (5.29) is easily evaluated:

$$U_1(t) = -\ln(1 - e^{-t}) \quad (5.32)$$

And we see that \tilde{U}_1 is given by

$$\tilde{U}_1 = -\ln(t) \quad (5.33)$$

From (5.33) and (5.31) it then follows that

$$\tilde{U}_p(t) = (-1)^p \frac{t^{p-1}}{(p-1)!} \ln(t) \quad (5.34)$$

Inserting this in (5.28) gives

$$F_s(\beta s) = -\frac{1}{4\pi} \left(t + \frac{t^2}{8} + \frac{t^3}{192} - \frac{t^4}{3072} + \dots \right) \ln(t) \quad (5.35)$$

Where $F_s(\beta s)$ is the singular part of the free energy and $t = 2 \ln(\cosh(2\beta s))$. Expanding (5.35) in powers of βs gives

$$F_s(\beta s) = -\frac{2}{\pi} \left[(\beta s)^2 - \frac{1}{6} (\beta s)^4 + \frac{23}{180} (\beta s)^6 - \frac{593}{5040} (\beta s)^8 + \dots \right] \ln|\beta s| \quad (5.36)$$

5.6 Perturbation theory

We now proceed with the derivation of a perturbation theory about the free fermion line of a 6-vertex model. The Hamiltonian of a general 6-vertex model can be defined as follows. One assigns an energy $e(p, i)$ to a vertex in state p (see fig. 5.1) and position i . The configuration of the lattice can be specified by a function c which maps a position of a vertex to a number, $1 \dots 6$, which is to be interpreted as the state of the vertex at that position. The reduced Hamiltonian (H) is defined to be the functional that assigns to each state c its energy times $-\beta$. We can thus write

$$H(c) = -\beta \sum_i e(c(i), i) \quad (5.37)$$

For H a Hamiltonian of a general 6-vertex model and H_0 a Hamiltonian of a free fermion model a perturbation V can be defined so that we have

$$H = H_0 + V \quad (5.38)$$

The partition function Z can be written as:

$$Z = \sum_c e^{H_0(c) + V(c)} = Z_0 \langle e^V \rangle \quad (5.39)$$

Here Z_0 is the partition function of the free fermion model. The reduced free energy can be expressed as:

$$F = F_0 + \ln \langle e^V \rangle = F_0 + \langle V \rangle + \frac{1}{2} \langle (V - \langle V \rangle)^2 \rangle + \dots \quad (5.40)$$

Here F_0 is the reduced free energy of the free fermion model. Now write $V = \sum_i V_i$ with $V_i(c(i))$ a perturbation of the vertex energy times $-\beta$ at position i . (5.40) can be rewritten as:

$$F = F_0 + \sum_i \langle V_i \rangle + \frac{1}{2} \sum_{ij} [\langle V_i V_j \rangle - \langle V_i \rangle \langle V_j \rangle] + \dots \quad (5.41)$$

To compute a free fermion average $\langle V_{i_1} V_{i_2} \dots V_{i_n} \rangle$, we can proceed as follows: Introduce a constraint in the free fermion model by requiring the vertices at the positions $i_1 \dots i_n$ to be in the states $x_1 \dots x_n$. The partition function of this model is denoted by $Z_{i_1 \dots i_n}(x_1 \dots x_n)$. We can then write

$$\langle V_{i_1} V_{i_2} \dots V_{i_n} \rangle = \sum_{x_1 \dots x_n} \frac{Z_{i_1 \dots i_n}(x_1 \dots x_n) V(x_1) \dots V(x_n)}{Z_0} \quad (5.42)$$

It now remains to calculate $Z_{i_1 \dots i_n}(x_1 \dots x_n)$. It is convenient to reformulate this problem as follows: Denote the state of an arrow located at the bond j by s_j . Put $s_j = 1$ if the arrow points oppositely to the ground state configuration and $s_j = 0$ otherwise. Define a constrained free fermion model by requiring the arrow at the bond j_r to be in state s_{j_r} for $1 \leq r \leq m$. We then want to evaluate the partition function of this model, which we denote as $Z(s_{j_1} \dots s_{j_m})$. The idea is to perturb the weights of the dimers on the bonds j_r infinitesimally. We assign a weight $C(1 + \epsilon_r)$ to the dimer on the bond j_r . The partition function

of the (unconstrained) free fermion model ($Z(\epsilon_1 \dots \epsilon_m)$) can be written in terms of the constrained partition functions as:

$$\begin{aligned} Z(\epsilon_1 \dots \epsilon_m) &= \sum_{\{s\}} Z(s_{j_1} \dots s_{j_m}) \prod_{k=1}^m (1 + s_{j_k} \epsilon_k) \\ &= Z_0 + \sum_k Z(s_{j_k} = 1) \epsilon_k + \sum_{k < l} Z(s_{j_k} = 1, s_{j_l} = 1) \epsilon_k \epsilon_l + \dots \end{aligned} \quad (5.43)$$

$Z(\epsilon_1 \dots \epsilon_m)$ can be calculated using (5.10), by making the necessary changes to R . We can write:

$$R = R_0 + \sum_{k=1}^m \epsilon_k R_{(k)} \quad (5.44)$$

Here R_0 is the original unperturbed matrix, $R_{(k)}$ is defined as follows:

$$\begin{aligned} R_{(k),ij} &= C && \text{if } i \text{ and } j \text{ are connected by } j_k \text{ and } i \text{ points to } j, \\ R_{(k),ij} &= -C && \text{if } i \text{ and } j \text{ are connected by } j_k \text{ and } j \text{ points to } i, \\ R_{(k),ij} &= 0 && \text{if } i \text{ and } j \text{ are not connected by } j_k. \end{aligned}$$

Note that the $R_{(k)}$ have only two nonzero matrix elements. Inserting (5.44) in (5.10) and expanding gives:

$$\begin{aligned} Z &= \sqrt{\det R} = \sqrt{\det R_0} e^{\frac{1}{2} \text{Tr} \ln [1 + \sum_k \epsilon_k R_0^{-1} R_{(k)}]} \\ &= \sqrt{\det R_0} \left[1 + \frac{1}{2} \sum_k \epsilon_k \text{Tr} (R_0^{-1} R_{(k)}) \right. \\ &\quad \left. + \frac{1}{4} \sum_{k,l} \epsilon_k \epsilon_l \left[\frac{1}{2} \text{Tr} (R_0^{-1} R_{(k)}) \text{Tr} (R_0^{-1} R_{(l)}) - \text{Tr} (R_0^{-1} R_{(k)} R_0^{-1} R_{(l)}) \right] + \dots \right] \end{aligned} \quad (5.45)$$

Using (5.45) and (5.43) we can directly read off the constrained partition functions which have all their arguments set to +1 (i.e. all the constrained arrows point oppositely to the ground state configuration). To calculate a constrained partition function with some of its arguments set to 0, we simply have to apply the principle of inclusion and exclusion (i.e. a Möbius inversion on the power set of arguments). For example consider the evaluation of $Z(s_1, s_2, s_3, s_4, s_5)$, with $s_1 = s_2 = 1$ and $s_3 = s_4 = s_5 = 0$. Put $t_3 = t_4 = t_5 = 1$. According to the principle of inclusion and exclusion, we can write:

$$\begin{aligned} Z(s_1, s_2, s_3, s_4, s_5) &= Z(s_1, s_2) \\ &\quad - [Z(s_1, s_2, t_3) + Z(s_1, s_2, t_4) + Z(s_1, s_2, t_5)] \\ &\quad + Z(s_1, s_2, t_3, t_4) + Z(s_1, s_2, t_3, t_5) + Z(s_1, s_2, t_4, t_5) \\ &\quad - Z(s_1, s_2, t_3, t_4, t_5) \end{aligned} \quad (5.46)$$

5.7 First order computation for the staggered F-model

For the staggered F-model the expansion can be simplified. The vertex in the ground state at a particular point will be referred to as an a-vertex. A b-vertex

is obtained by reversing the arrows of an a-vertex. An a-vertex (b-vertex) is thus of type 5 or 6 and has an energy of $-s$ (s). The constrained partition function corresponding to the model with one vertex constrained to be an a-vertex (b-vertex) is denoted as Z_a (Z_b). Note that under the transformation $s \rightarrow -s$ the rôle of vertices a and b are interchanged. We thus have

$$Z_a(\beta s) = Z_b(-\beta s) \quad (5.47)$$

If we put $\beta\epsilon = \frac{1}{2} \ln(2) + U$ we have, according to (5.41) and (5.42), to first order in U :

$$F = F_0 - \frac{Z - Z_a - Z_b}{Z_0} U + O(U^2) \quad (5.48)$$

Here F is the reduced free energy per vertex of the staggered F-model. To calculate Z_b we only have to constrain two opposing arrows of one vertex to point oppositely to an a-vertex. If we choose to constrain two horizontal arrows, we need the R_{ij}^{-1} for i and j corresponding to an a or b variable on the decorated lattice. We define Green's functions $G_{x^{(i)}, x'^{(j)}}(p, q)$ with $x^{(i)} \in \{a, b, c, d\}$ and $x'^{(j)} \in \{a', b', c', d'\}$, so that the solution of (5.20) can be written as

$$x_{p,q}^{(i)} = \sum_j G_{x^{(i)}, x'^{(j)}}(p, q) x'^{(j)} \quad (5.49)$$

Solving equation (5.20) yields

$$\begin{aligned} G_{a,a'}(p, q) &= -\frac{1}{\Delta(p, q)} \frac{e^{\frac{1}{2}\beta s}}{2} (\beta^p - \beta^{-p}) \\ G_{a,b'}(p, q) &= -\frac{1}{\Delta(p, q)} \left[\frac{e^{\frac{1}{2}\beta s}}{2} (\beta^p + \beta^{-p}) + e^{-\frac{3}{2}\beta s} \alpha^q \right] \\ G_{b,a'}(p, q) &= \frac{1}{\Delta(p, q)} \left[\frac{e^{\frac{1}{2}\beta s}}{2} (\beta^p + \beta^{-p}) + e^{-\frac{3}{2}\beta s} \alpha^{-q} \right] \\ G_{b,b'}(p, q) &= \frac{1}{\Delta(p, q)} \frac{e^{\frac{1}{2}\beta s}}{2} (\beta^p - \beta^{-p}) \end{aligned} \quad (5.50)$$

An inverse Fourier transformation yields in the limit $N, M \rightarrow \infty$:

$$\begin{aligned} G_{a,a'}(n, m) &= -\frac{e^{\frac{1}{2}\beta s}}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \frac{e^{im\theta_1} (e^{i(n+1)\theta_2} - e^{i(n-1)\theta_2})}{\Delta(\theta_1, \theta_2)} \\ G_{a,b'}(n, m) &= -\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \left[\frac{\frac{e^{\frac{1}{2}\beta s}}{2} e^{im\theta_1} (e^{i(n+1)\theta_2} + e^{i(n-1)\theta_2})}{\Delta(\theta_1, \theta_2)} \right. \\ &\quad \left. + \frac{e^{-\frac{3}{2}\beta s} e^{i(m+1)\theta_1} e^{in\theta_2}}{\Delta(\theta_1, \theta_2)} \right] \\ G_{b,a'}(n, m) &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \left[\frac{\frac{e^{\frac{1}{2}\beta s}}{2} e^{im\theta_1} (e^{i(n+1)\theta_2} + e^{i(n-1)\theta_2})}{\Delta(\theta_1, \theta_2)} \right. \\ &\quad \left. + \frac{e^{-\frac{3}{2}\beta s} e^{i(m-1)\theta_1} e^{in\theta_2}}{\Delta(\theta_1, \theta_2)} \right] \\ G_{b,b'}(n, m) &= \frac{e^{\frac{1}{2}\beta s}}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \frac{e^{im\theta_1} (e^{i(n+1)\theta_2} - e^{i(n-1)\theta_2})}{\Delta(\theta_1, \theta_2)} \end{aligned} \quad (5.51)$$

Here $\Delta(\theta_1, \theta_2)$ is given by

$$\Delta(\theta_1, \theta_2) = 2 \cosh(2\beta s) + 2 \cos(\theta_1) \cos(\theta_2) \quad (5.52)$$

The necessary components of the matrix R^{-1} can be expressed in terms of the Green's function G as:

$$R_{x_n, m, x'_{n'}, m'}^{-1} = G_{x, x'}(n - n', m - m') \quad (5.53)$$

Put $W_b = \frac{Z_b}{Z_0}$. It follows from (5.43) and (5.45) that

$$W_b = \frac{1}{4} \text{Tr}^2(R_0^{-1} R_{(k)}) - \frac{1}{2} \text{Tr}(R_0^{-1} R_{(k)} R_0^{-1} R_{(l)}) \quad (5.54)$$

where k and l refer to the bonds that connect the points $a_{n, m}$ and $b_{n, m+1}$ respectively $a_{n, m+1}$ and $b_{n, m+2}$ on the decorated lattice. Using (5.7) the first term in (5.54) (W_{b1}) can be written as

$$W_{b1} = \frac{e^{-\beta s}}{4} [G_{b, a}(0, 1) - G_{a, b}(0, -1)]^2 \quad (5.55)$$

Inserting the appropriate expressions given in (5.51) in this equation yields

$$W_{b1} = \frac{1}{64\pi^4} \left[\int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \frac{e^{-2\beta s} + \cos(\theta_1) \cos(\theta_2)}{\cosh(2\beta s) + \cos(\theta_1) \cos(\theta_2)} \right]^2 \quad (5.56)$$

The second contribution to Z_b (W_{b2}) can be expressed in terms of the Green's function G as:

$$W_{b2} = -\frac{e^{-\beta s}}{2} [G_{ba}(0, 2) G_{ba}(0, 0) + G_{ab}(0, 0) G_{ab}(0, -2) - G_{aa}(0, 1) G_{bb}(0, -1) - G_{aa}(0, -1) G_{bb}(0, 1)] \quad (5.57)$$

Using (5.51) it is not difficult to see that each of the four terms in the brackets of this equation vanishes. We thus have

$$W_b = W_{b1} = \frac{1}{64\pi^4} \left[\int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \frac{e^{-2\beta s} + \cos(\theta_1) \cos(\theta_2)}{\cosh(2\beta s) + \cos(\theta_1) \cos(\theta_2)} \right]^2 \quad (5.58)$$

From (5.47) it follows that

$$W_a(\beta s) = W_b(-\beta s) \quad (5.59)$$

The expression obtained by substituting (5.58) and (5.59) in (5.48) can be simplified by exploiting the fact that we only need the even part of the function $W_b(\beta s)$. We will denote even (odd) parts of functions with a subscript $+$ ($-$). (5.48) can be rewritten as

$$F = F_0 + (2W_{b+} - 1)U + \dots \quad (5.60)$$

(5.58) can be rewritten as

$$W_b = I^2 \quad (5.61)$$

with

$$I = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \frac{e^{-2\beta s} + \cos(\theta_1) \cos(\theta_2)}{\cosh(2\beta s) + \cos(\theta_1) \cos(\theta_2)} \quad (5.62)$$

From (5.61) it follows that

$$W_{b+} = I_+^2 + I_-^2 \quad (5.63)$$

From (5.62) it easily follows that

$$I_+ = \frac{1}{2} \quad (5.64)$$

and

$$I_- = -\frac{1}{2} \frac{\partial F_0}{\partial \beta s} \quad (5.65)$$

(5.60) can thus be written as

$$F = F_0 + \frac{1}{2} \left[\left(\frac{\partial F_0}{\partial \beta s} \right)^2 - 1 \right] U + \dots \quad (5.66)$$

5.8 Singular behavior in the vicinity of the free fermion line

Equation (5.66) allows us to check the dependence of the Gaussian coupling j on $\beta\epsilon$ as given by (4.9) for $\beta\epsilon \approx \frac{1}{2} \ln(2)$. We have seen that renormalization group arguments lead to the following leading singularity in the reduced free energy:

$$F_s \sim (\beta s)^{\frac{2}{2-\frac{4}{d_f}}} \quad (5.67)$$

with

$$j = \frac{1}{2} \arccos \left(1 - \frac{1}{2} e^{2\beta\epsilon} \right) \quad (5.68)$$

If we put

$$\beta\epsilon = \frac{1}{2} \ln(2) + U \quad (5.69)$$

in these equations, we find that

$$F_s = A(U) (\beta s)^2 \left[-\frac{8}{\pi} (U + O(U^2)) \ln(\beta s) + \frac{32}{\pi^2} (U^2 + O(U^3)) \ln^2 |\beta s| + \dots \right] \quad (5.70)$$

If we compare this with (5.36), we find that the amplitude $A(U)$ is given by

$$A(U) = \frac{1}{4U} \quad (5.71)$$

It then follows that the amplitude of the term $(\beta s)^2 \ln^2 |\beta s|$ is $\frac{8}{\pi^2} (U + O(U^2))$. It is now a simple matter to verify this using (5.66) and (5.36). From (5.36) and (5.66) it follows that the order U contribution to the singular part of the reduced free energy $F_1(\beta s)$ can be written as

$$F_{1,s}(\beta s) = [B_1(\beta s) \ln |\beta s| + B_2(\beta s) \ln^2 |\beta s|] U \quad (5.72)$$

with B_1 and B_2 regular functions of βs . Inserting (5.36) in (5.66) gives

$$B_2(\beta s) = \frac{8}{\pi^2} \left[(\beta s)^2 - \frac{2}{3} (\beta s)^4 + \frac{79}{90} (\beta s)^6 + \dots \right] \quad (5.73)$$

We have thus verified (5.68) to first order in U .

5.9 Linked cluster expansion

In this section we will derive a diagrammatic (linked cluster) method to compute terms in the perturbative expansion of the free energy. Instead of perturbing the energies of vertices 1...4, we will perturb the energy of an a-vertex. We will consider the model with the following vertex energies: vertices 1...4 are assigned an energy of ϵ_0 , with $\beta\epsilon_0 = \frac{1}{2} \ln(2)$, a-vertices are assigned an energy of $-s_0 + v$, and b-vertices are assigned an energy of s_0 . The reduced free energy of this model is denoted as $\tilde{F}(-\beta v)$. It is not difficult to see that the reduced free energy of the staggered F-model can be obtained from \tilde{F} :

$$F\left(\beta\epsilon = \beta\epsilon_0 + \frac{1}{2}U, \beta s = \beta s_0 + \frac{1}{2}U\right) = \tilde{F}(U) - \frac{1}{2}U \quad (5.74)$$

We now define a Hamiltonian $H(\{J\})$ as

$$H(\{J\}) = H_0 + \sum_i J_i V_i \quad (5.75)$$

H_0 is a free fermion model and the V_i are defined as follows. If c is an arrow configuration then $V_i(c)$ is the increase in energy times $-\beta$ of the vertex at position i (c_i). In this case we thus have $V_i(c) = U$ if c_i is an a-vertex and $V_i(c) = 0$ otherwise. If we put all the $J_i = 1$, $H(\{J\})$ becomes the Hamiltonian of the model defined above. The partition function for $H(\{J\})$ is denoted as $Z(\{J\})$. In the following we shall use the notation $J = p$ to indicate that $J_i = p$ for all i . An expansion about $J = 0$ yields

$$Z(J = 1) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1 \dots i_k} \left. \frac{\partial^k Z(\{J\})}{\partial J_{i_1} \dots \partial J_{i_k}} \right|_{J=0} \quad (5.76)$$

From the definition of $Z(\{J\})$ it follows that the free fermion correlations $g(i_1 \dots i_k)$, defined as

$$g(i_1, i_2 \dots i_k) \equiv \langle V_{i_1} V_{i_2} \dots V_{i_k} \rangle \quad (5.77)$$

can be expressed as

$$g(i_1, i_2 \dots i_k) = \frac{1}{Z(J=0)} \left. \frac{\partial^k Z(\{J\})}{\partial J_{i_1} \dots \partial J_{i_k}} \right|_{J=0} \quad (5.78)$$

The terms in the expansion (5.76) are thus readily expressed in terms of the free fermion correlations. We have

$$Z(J = 1) = Z(J = 0) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1 \dots i_k} g(i_1 \dots i_k) \quad (5.79)$$

To compute the free fermion correlations we define for $i_1 \cdots i_k$ k points on the lattice, a constrained free fermion model by demanding that the vertices at positions $i_1 \cdots i_k$ be a -vertices. The partition function of this model is denoted as $Z_{i_1 \cdots i_k}$. In terms of $Z_{i_1 \cdots i_k}$, $g(i_1 \cdots i_k)$ can be written as

$$g(i_1 \cdots i_k) = \frac{Z_{i_1 \cdots i_k}}{Z(J=0)} U^k \quad (5.80)$$

From the correspondence of arrow configurations to dimer configurations, it follows that $Z_{i_1 \cdots i_k}$ can be written as the sum of all dimer configurations such that at the cities $i_1 \cdots i_k$ dimers are placed on two internal bonds. For each of these cities there are two choices for their internal dimer configurations. The sum of all dimer configurations consistent with one particular choice at each city is denoted as $Z'_{i_1 \cdots i_k}$. Since $Z'_{i_1 \cdots i_k}$ does not depend on the particular choice made, it is clear that we have

$$Z_{i_1 \cdots i_k} = 2^r Z'_{i_1 \cdots i_k} \quad (5.81)$$

where r is the number of different indices in $i_1 \cdots i_k$. For definiteness we assume that at each city i_p , for $p = 1 \cdots k$ dimers are placed on respectively the bond connecting the points labeled by the variables a_{i_p} and c_{i_p} , and d_{i_p} and b_{i_p} (see fig. 5.3). These bonds will be referred to as modified bonds, and the position of these two bonds will be denoted as, respectively, i_{p_1} and i_{p_2} . Dimers on modified bonds will be referred to as modified dimers, and the modified lattice is defined to be the decorated lattice, but with only the modified bonds. To compute $Z_{i_1 \cdots i_k}$, we define an inhomogeneous free fermion model by changing the weights of the modified bonds. At all positions j we associate the variables ϵ_{j_1} and ϵ_{j_2} to, respectively, the modified bonds at j_1 and j_2 . We now choose the weight of a dimer on a modified bond at a position k to be a function of ϵ_k , so that we have

$$Z_{i_1 \cdots i_k} = \text{coeff. of } \epsilon_{i_{1_1}} \epsilon_{i_{1_2}} \cdots \epsilon_{i_{k_1}} \epsilon_{i_{k_2}} \text{ in } Z(\{\epsilon\}) \quad (5.82)$$

Here $Z(\{\epsilon\})$ is the partition function of the inhomogeneous free fermion model. We denote the weight of the dimer on a modified bond at a position k as $w(\epsilon_k)$. The choice $w(\epsilon_k) = u(1 + \sqrt{2}\epsilon_k)$ (u is the original weight of the dimer) seems to work, but there is a problem if some of the indices in (5.82) coincide. To see this, consider the r.h.s. of (5.82), when all the indices in this equation are chosen differently. It is clear that a dimer configuration contributing to $Z(\{\epsilon\})$ has a weight proportional to $\epsilon_{i_{1_1}} \epsilon_{i_{1_2}} \cdots \epsilon_{i_{k_1}} \epsilon_{i_{k_2}}$ only if it consists of all the modified dimers at the positions $i_{1_1}, i_{1_2}, \dots, i_{k_1}, i_{k_2}$. The r.h.s. of (5.82) is thus equal to $2^k Z'_{i_1 \cdots i_k}$, and this equals according to (5.81) the r.h.s. of (5.82). It is also clear that with the present choice of the dimer weights $Z(\{\epsilon\})$ is a multilinear function of the ϵ 's, so (5.82) is certainly not true when not all of the indices are different. Although this doesn't seem to be a big problem, because there is no point in constraining a certain vertex more than once, we have to sum over all indices to calculate $Z(J=1)$ and it is convenient to do so unrestrictively. The solution to this problem is obvious. We define $w(\epsilon_k)$ as

$$w(\epsilon_k) = u \left(1 + \frac{\sqrt{2}\epsilon_k}{1 - \epsilon_k} \right) \quad (5.83)$$

Since $Z(\{\epsilon\})$ is a free fermion model, we can use (5.10):

$$Z(\{\epsilon\}) = \sqrt{\det R(\{\epsilon\})} \quad (5.84)$$

with

$$R(\{\epsilon\}) = R(\epsilon = 0) + \sum_k R_{(k)} \quad (5.85)$$

where the sum is over all modified bonds, $\epsilon = 0$ means $\epsilon_i = 0$ for all i and $R_{(k)}$ is defined as follows. If the bond k connects the points i to j we put

$$R_{(k)ij} = \pm \sqrt{2} \frac{u\epsilon_k}{1 - \epsilon_k} \quad (5.86)$$

The plus sign is chosen if the orientation on the bond is such that i points to j , else the minus sign is chosen. All other matrix elements of $R_{(k)}$ are zero. (5.84) can be rewritten as

$$Z(\{\epsilon\}) = Z_0 e^{\frac{1}{2} \text{Tr} \ln(1+GB)} \quad (5.87)$$

where $G = R^{-1}(\epsilon = 0)$, $Z_0 = Z(\{\epsilon = 0\})$ and $B = \sum_k R_{(k)}$. Note that B_{ij} is only nonzero if i and j are connected by a modified bond, in which case

$$B_{ij} = R_{(k)ij} \quad (5.88)$$

if the modified bond k connects i to j . Expanding the logarithm in this equation allows us to rewrite the exponent of the above equation as

$$\frac{1}{2} \text{Tr} \ln(1 + GB) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{i_1, j_1 \dots i_n, j_n} \prod_{r=1}^n G_{i_r, j_{r+1}} B_{j_{r+1}, i_{r+1}} \quad (5.89)$$

Here $i_{n+1} = i_1$ and $j_{n+1} = j_1$. Note that for a contribution to the summation on the r.h.s. of this equation to be nonzero, i_{r+1} must be chosen so that j_{r+1} and i_{r+1} are connected by a modified bond. The summation can then be interpreted as a summation of amplitudes of all closed paths of n steps, where a single step consists of going from a position i_r to an arbitrary position j_{r+1} , and then from j_{r+1} to that point (i_{r+1}) which is connected to j_{r+1} by a modified bond. If $r = n$, j_{r+1} is, of course, not arbitrary because of the conditions $i_{n+1} = i_1$ and $j_{n+1} = j_1$. Hence it follows that we are dealing with closed paths. To each step we associate a weight which in the example given above equals $G_{i_r, j_{r+1}} B_{j_{r+1}, i_{r+1}}$. The amplitude of a path is the product of the weights of the steps forming the path. Each closed path can be defined as an oriented loop of which one point plays the rôle of starting and end point. It is clear that the amplitude of the path does not depend on the choice of the starting point. It also does not depend on the orientation, because both G and B are antisymmetric. Instead of summing over closed paths, we may thus sum over loops. To do this, we need to know the number of closed paths that correspond to one loop. It is clear that this is given by two times the number of ways one can attach a starting point to the loop. If the loop consists of n steps, and if the winding number of the loop is p (i.e. traversing the loop once, means visiting the same points in the same order p times) then there are $\frac{n}{p}$ ways of

attaching an starting point to the loop. Suppose all loops are enumerated in some arbitrary order. We define the amplitude of the r^{th} loop, denoted as L_r , as the amplitude of one of the closed paths to which it corresponds divided by the winding number of the loop and multiplied by a sign (± 1). This sign is the product of the following factors. For each modified bond there is a factor -1 , and there is an overall factor of -1 . We can now rewrite (5.89) as

$$\frac{1}{2} \text{Tr} \ln (1 + GB) = \sum_{r=1}^{\infty} L_r \quad (5.90)$$

Inserting this in (5.87) gives

$$\frac{Z(\{\epsilon\})}{Z_0} = \prod_{r=1}^{\infty} e^{L_r} \quad (5.91)$$

It thus follows that $\frac{Z(\{\epsilon\})}{Z_0}$ is a sum of amplitudes of diagrams, where a diagram is a set of loops and the amplitude of a diagram is the product of amplitudes of the loops of which it consists multiplied by a factor $\prod_{r=1}^{\infty} \frac{1}{k_r!}$ if loop r occurs k_r times in the diagram. Note that with our original definition of the weight function $w(\epsilon) = u(1 + \sqrt{2}\epsilon)$, $Z(\{\epsilon\})$ would be a multilinear function of the ϵ 's. This implies that products of loops sharing bonds will not make a net contribution (this does not, of course, depend on the choice of the weight function). To see an example of this consider two different loops with amplitudes L_1 and L_2 sharing a bond. In that case there is a contribution $L_1 L_2$ to $Z(\{\epsilon\})$. There is also a single loop, obtained by merging the two loops. This loop has an amplitude of $-L_1 L_2$, and we see that the two contributions have indeed cancelled. Similarly, contributions from loops with winding numbers greater than one cancel against products of loops with smaller winding numbers. Although, as we shall see later, such cancellations will also occur in the expansion of $Z(J=1)$, $\ln(Z(J=1))$ will contain only connected diagrams (the precise definition of “connected” will be given later). Diagrams in which a bond appears more than once can make a net contribution to $\ln(Z(J=1))$ because the disconnected diagrams against which such a contribution would cancel don't appear in the expansion of $\ln(Z(J=1))$. To take into account the cancellation of loops with a winding number greater than one against products of loops with lower winding numbers, we can rewrite (5.91) as

$$\frac{Z(\{\epsilon\})}{Z_0} = \prod_{r=1}^{\infty} [1 + L_r] \quad (5.92)$$

Here the product is only over loops with a winding number of one. To derive this equation directly from (5.90) one proceeds as follows. The amplitude of a loop is calculated as above, but without the overall minus sign. We may then write the amplitude of a loop with winding number p as

$$A_{r,p} = \frac{A_r^p}{p} \quad (5.93)$$

Here $A_{r,p}$ is the amplitude of the loop with winding number p and A_r is the “same” loop with winding number one. We can then rewrite (5.90) as

$$\frac{1}{2} \text{Tr} \ln (1 + GB) = - \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} \frac{A_r^p}{p} = \sum_{r=1}^{\infty} \ln (1 - A_r) \quad (5.94)$$

Here the sum over r is only over loops with a winding number of one. Inserting this in (5.87) then gives (5.92) (In that equation L_r is defined as $-A_r$). With our definition of the weight function (5.83) a diagram contributing to $Z(\{\epsilon\})$ is a multinomial in the ϵ 's. To have that each diagram is a monomial, we split each modified bond on the decorated lattice into an infinite number of bonds, each of them connecting the same two points as the original bond. These bonds are all assigned the same orientation as the original bond. The weights of these bonds are chosen by assigning to each bond a (different) term in the expansion of (5.83) in powers of ϵ . It now follows that (5.91) still holds on this new lattice. We can see this as follows. Let all loops on the new lattice be enumerated. The amplitude of the k^{th} loop will be denoted as \tilde{L}_k . Since the sum of the weights of the new bonds replacing one modified bond equals the weight of the modified bond, we can write

$$\sum_{r=1}^{\infty} L_r = \sum_{k=1}^{\infty} \tilde{L}_k \quad (5.95)$$

(5.91) can thus be rewritten as

$$\frac{Z(\{\epsilon\})}{Z_0} = \prod_{r=1}^{\infty} e^{L_r} = e^{\sum_{r=1}^{\infty} L_r} = e^{\sum_{k=1}^{\infty} \tilde{L}_k} = \prod_{k=1}^{\infty} e^{\tilde{L}_k} \quad (5.96)$$

Note that the cancellation of diagrams discussed above implies that a diagram on the new lattice which uses more than bond connecting the same points will make no net contribution. In particular loops with winding numbers greater than one will cancel against products of loops with smaller winding numbers. To emphasize this we can rewrite (5.92) as follows. We say that a loop on the new lattice corresponds to a loop on the modified lattice if the same points can be traversed in the same order. The amplitude of a loop on the new lattice corresponding to a loop r on the modified lattice (both with a winding number of one) shall be written as $\tilde{L}_{r,k}$, where the index k enumerates all such loops. We can then write

$$L_r = \sum_{k=1}^{\infty} \tilde{L}_{r,k} \quad (5.97)$$

Inserting this in (5.92) then gives

$$\frac{Z(\{\epsilon\})}{Z_0} = \prod_{r=1}^{\infty} \left[1 + \sum_{k=1}^{\infty} \tilde{L}_{r,k} \right] \quad (5.98)$$

We shall use the following notations for the bonds in a city. For a city at position i , there is a bond with a weight proportional to $\epsilon_{i_1}^k$ and $\epsilon_{i_2}^k$, if $k \geq 1$. These two bonds will be referred to as bonds of order k . A bond that connects the points a_i and c_i will be referred to as a bond of type 1, a bond that connects the points b_i and d_i will be referred to as a bond of type 2. A diagram will be called balanced if at every city the sum of the orders of bonds of type 1 in the diagram equals the sum of the orders of bonds of type 2. From (5.82) it follows that only balanced diagrams contribute to $Z_{i_1 \dots i_n}$. It is also clear that a particular balanced diagram contributes to $Z_{i_1 \dots i_n}$ only if at each city the sum

of the orders of bonds of each type equals the number of times the city occurs as an index in $Z_{i_1 \dots i_n}$. For balanced diagrams the sum of the orders of bonds of a certain type at a city will be referred to as the order of the diagram at that city. If a city is not part of a diagram, then the order of that diagram at that city is zero. From (5.80) it now follows that $g(i_1 \dots i_n)$ can be written as the sum of amplitudes of balanced diagrams, if the weights associated with bonds are redefined as follows. We define the weights associated with the modified bonds, by choosing $\epsilon = \sqrt{U}$, so that a bond of order k has a weight of $\pm\sqrt{2} \left(\sqrt{U}\right)^k$. $g(i_1 \dots i_n)$ can thus be written as a restricted sum of amplitudes of balanced diagrams, where the restriction implies that a diagram contributes if at each city the order of the diagram equals the number of times the city occurs as an argument of g . (5.79) expresses $Z(J=1)$ as

$$Z(J=1) = Z(J=0) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1 \dots i_k} g(i_1 \dots i_k) \quad (5.99)$$

By substituting the diagrammatic expansion of g in this equation, we see that $\frac{Z(J=1)}{Z(J=0)}$ is a sum of balanced diagrams. We denote the order of a balanced diagram at a city j as m_j . We now focus on a balanced diagram with $\sum_j m_j = k$. This diagram will appear $\frac{k!}{\prod_j m_j!}$ times in (5.79). The contribution this diagram makes to $\frac{Z(J=1)}{Z(J=0)}$ is thus given by its amplitude multiplied by the factor $\frac{1}{\prod_j m_j!}$.

The fact that this extra factor appears is rather inconvenient. This problem can be overcome by again changing the weights associated with the bonds. Since diagrams which use more than one bond of a certain type at a city cancel, we can absorb a factor $\frac{1}{\sqrt{r!}}$ in the weight of a bond of order r . The weight of a bond of order r now becomes $\pm\sqrt{2} \frac{(\sqrt{U})^r}{\sqrt{r!}}$. With this definition of the weights of the bonds $\frac{Z(J=1)}{Z(J=0)}$ is thus simply the sum of amplitudes of all balanced diagrams. Note that the factor $\frac{1}{\sqrt{r!}}$ is rather arbitrary, we could have chosen $\frac{1}{(r!)^p}$ for the weight of a bond of order r of one type and $\frac{1}{(r!)^q}$ for the bond of order r of the other type, provided that $p+q=1$. This arbitrariness shall lead to a drastic simplification for the expansion of the free energy.

We now proceed to write $\frac{Z(J=1)}{Z(J=0)}$ in a way analogous to the expression (5.96) for $\frac{Z(\{\epsilon\})}{Z_0}$, so that it becomes a simple matter to find the diagrammatic expansion of $\ln(Z(J=1))$. We can do this as follows. We define a diagram to be connected (irreducible), if it is balanced and cannot be written as a product of smaller balanced diagrams (by a smaller diagram we mean a diagram that contains only a part of the loops of the original diagram). Suppose all connected diagrams are enumerated in some arbitrary order. The amplitude of the r^{th} connected diagram is denoted as C_r . We can now express $\frac{Z(J=1)}{Z(J=0)}$ in terms of the C_r as

$$\frac{Z(J=1)}{Z_0} = \prod_{r=1}^{\infty} [1 + C_r] \quad (5.100)$$

In this equation the product is over connected diagrams consisting of loops with winding numbers of one. The reduced free energy of the staggered F-model can

thus be written as

$$\tilde{F} = F_0 + \sum_{r=1}^{\infty} \ln(1 + C_r) \quad (5.101)$$

As noted earlier, there is some freedom in the choice of the weights of the bonds. At a certain city, only the product of the weights of the two bonds of order r is required to contain the factor $\frac{1}{r!}$. For the expansion of the free energy this means that a diagram that uses a bond of order r of one type also has to contain the bond of order r of the other type, otherwise the diagram will cancel against other diagrams. We can thus go back to the modified lattice by summing over the orders of the bonds. The weight of a modified bond then becomes $\pm\sqrt{2(\exp(u) - 1)}$. The sign is given according to the orientation of the bond. The extra minus sign can now be omitted, because any diagram has an even number of bonds.

Appendix A

Notations and conventions

A.1 Vectors

Vectors are denoted like ordinary variables (i.e. without an arrow). The modulus of a vector x is denoted as $|x|$. By x^n with n an even integer we mean $|x|^n$.

A.2 Summation convention

Repeated indices are summed over, unless stated otherwise.

A.3 Multi-indices

A multi-index containing n indices is denoted as (n) . We use a summation convention in the following way:

$$A_{(n)}B_{(n)} = A_{i_1 \dots i_n} B_{i_1 \dots i_n} \quad (\text{A.1})$$

A possible sum over n is always explicitly denoted. Different variables between brackets always contain different indices. e.g.

$$A_{(n)}B_{(m)} = A_{i_1 \dots i_n} B_{j_1 \dots j_m} \quad (\text{A.2})$$

Identical variables between brackets contain identical indices (as in (A.1)). A tensor can have more than one multi-index:

$$A_{(n),(m)} = A_{i_1 \dots i_n, j_1 \dots j_m} \quad (\text{A.3})$$

By

$$\sum_{\text{contractions}} A_{(n_1),(n_2) \dots (n_r)} \quad (\text{A.4})$$

we mean a sum over all possible contractions of the indices with each other, e.g.

$$\sum_{\text{contractions}} A_{(n_1),(n_2)} \quad (\text{A.5})$$

with $n_1 = n_2 = 2$ stands for

$$A_{iikk} + A_{ikik} + A_{ikki} \quad (\text{A.6})$$

A.4 Derivatives

For derivatives we sometimes use the notation $\partial_{(k),b}^{(a)}$. By this we mean a the k^{th} derivative with respect to the components (k) of b evaluated at the point a . The arguments a and b are optional.

A.5 Distributions

For a detailed account of distributions we refer to [31, 32]. Once a well defined space of functions S is defined, a space of continuous linear functionals S' on S can be defined. We shall define the set S as the set of all functions $h(x)$ that can be written as in (3.8). Apart from the usual L^2 -norm we define a Fréchet norm as follows:

$$|h| = \sup_{(k), x, k \leq N} |\partial_{(k)} h(x)| \quad (\text{A.7})$$

Here N is some arbitrary number. The set of all continuous linear functionals (distributions) with respect to this norm is denoted by S'_N . The action of a distribution $T \in S'_N$ on $h \in S$ is denoted as Th or as $T(h)$. It is not difficult to see that the functional T defined as

$$Th = \partial_{(n)} h|_z \quad (\text{A.8})$$

for some multi-index (n) is an element of S'_N when $n \leq N$. The functional I defined as

$$Ih = \int I(x) h(x) d^2 x \quad (\text{A.9})$$

also defines a distribution as long as $I(x)$ is bounded. A weak-topology on S'_N is defined by saying that $T_n \rightarrow T$ if $T_n h \rightarrow Th \forall h \in S$.

Appendix B

Gaussian correlations

B.1 Correlation function for $h^{(2)}$

In this appendix we derive the expression (3.27) for the correlation function for the field $h^{(2)}$. By definition we have:

$$G(x) = \left\langle h^{(2)}(x) h^{(2)}(0) \right\rangle \quad (\text{B.1})$$

In terms of \hat{h} this becomes

$$G(x) = \frac{1}{V} \sum_{k_1, k_2} \langle h(k_1) h(k_2) \rangle e^{ik_1 \cdot x} \quad (\text{B.2})$$

where the sum is over elements of the set $S^{(2)}$ (see section 3.3 for the definition of this set). Since in the Gaussian model $\hat{h}(k_1)$ and $\hat{h}(k_2)$ are uncorrelated unless $k_1 = \pm k_2$, we have

$$G(x) = \frac{1}{V} \sum_{k \in S^{(2)}} \left\langle \left| \hat{h}(k) \right|^2 + \left(\hat{h}(k) \right)^2 \right\rangle e^{ik \cdot x} \quad (\text{B.3})$$

Now define real valued variables $a(k)$ and $b(k)$ as follows:

$$h(k) = a(k) + ib(k) \quad (\text{B.4})$$

Since $h(x)$ is real we have

$$h(-k) = a(k) - ib(k) \quad (\text{B.5})$$

From (B.4) and (B.5) it follows that

$$dh(k) dh(-k) \equiv 2da(k) db(k) \quad (\text{B.6})$$

The partition function of the Gaussian model can be written as

$$Z = \prod_{k \in S^{(2)}} ' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} da(k) db(k) N(k) e^{-j(k)k^2(a(k)^2+b(k)^2)} = \prod_{k \in S^{(2)}} ' \frac{\pi N(k)}{j(k)k^2} \quad (\text{B.7})$$

The prime indicates the condition $k_y > 0$ and we have replaced the Gaussian coupling j by a k -dependent coupling $j(k)$. $N(k)$ is an uninteresting function coming from the definition of the measure Dh (see (3.14)), and the Jacobian of the transformation to the variables $a(k)$ and $b(k)$. It is clear from (B.7) that $\left\langle \left(\hat{h}(k) \right)^2 \right\rangle$ vanishes. $\left\langle \left| \hat{h}(k) \right|^2 \right\rangle$ is computed as follows:

$$\left\langle \left| \hat{h}(k) \right|^2 \right\rangle = -\frac{1}{k^2} \frac{\partial \ln(Z)}{\partial j(k)} \Big|_{j(k)=j} = \frac{1}{jk^2} \quad (\text{B.8})$$

Using this one can write (B.3) as:

$$G(x) = \frac{1}{V} \sum_{k \in S^{(2)}} \frac{e^{ik \cdot x}}{jk^2} \quad (\text{B.9})$$

We can formally rewrite (B.9) using the characteristic function $P_c^{(2)}$ of the set $S^{(2)}$ as:

$$G(x) = \int \frac{d^2 k}{(2\pi)^2} \frac{P^{(2)}(k)}{jk^2} e^{ik \cdot x} \quad (\text{B.10})$$

B.2 The height-height correlation function

In this section we will evaluate the height-height correlation function $(R(x))$ for the Gaussian model. By definition we have

$$R(x) = \left\langle (h(x) - h(0))^2 \right\rangle \quad (\text{B.11})$$

Using (3.27) we can write this as

$$R(x) = -\frac{2}{j} \int \frac{d^2 k}{(2\pi)^2} \frac{e^{ik \cdot x} - 1}{k^2} P(k) \quad (\text{B.12})$$

Integrating over the angle between k and x yields

$$R(x) = -\frac{1}{\pi j} \int_0^\infty \frac{d|k|}{|k|} (J_0(|k||x|) - 1) P(k) \quad (\text{B.13})$$

Differentiating both sides of this equation with respect to $|x|$ yields

$$R'(x) = -\frac{1}{\pi j |x|} \int_0^\infty d|k| J_0'(|k||x|) P(k) = -\frac{1}{\pi j |x|} \int_0^\infty dt J_0'(t) P\left(\frac{t}{|x|}\right) \quad (\text{B.14})$$

where we have substituted $t = |k||x|$. For large x the integral is -1 . We thus have

$$R(x) \sim \frac{1}{\pi j} \ln(|x|) \quad (\text{B.15})$$

Appendix C

Calculation of $\langle e^{iTh} \rangle$

In this appendix we derive equation (3.45). First we derive this equation in the special case of a distribution T that can be written as a finite linear combination of Dirac delta's. By taking appropriate limits of this special case we arrive at (3.45).

Let the distribution T be defined by:

$$Th = \sum_n \alpha_n h(x_n) \quad (\text{C.1})$$

In this case we have

$$\langle \exp(iTh) \rangle = \exp(iTh^{(1)}) \langle \exp(iTh^{(2)}) \rangle \quad (\text{C.2})$$

Using (3.23) we get

$$\begin{aligned} \langle \exp(iTh^{(2)}) \rangle &= \int Dh^{(2)} \exp(i \sum_n \alpha_n h^{(2)}(x_n)) \\ &\times \exp\left(-\frac{i}{2} \int (\nabla h^{(2)})^2 d^2x\right) \end{aligned} \quad (\text{C.3})$$

In terms of $h(k)$ this can be written as

$$\int Dh^{(2)} \exp\left(-\sum_k \left[\frac{j}{2} k^2 |h(k)|^2 - i \sum_n \frac{1}{\sqrt{V}} h(k) \exp(ik \cdot x_n)\right]\right) \quad (\text{C.4})$$

Here the sum is over those k -values that belong to $h^{(2)}$. We write $h(k) = a(k) + ib(k)$, with $a(k)$ and $b(k)$ real. The integral (C.4) can be written as:

$$\begin{aligned} &\left[\prod_k' \int 2da(k) db(k) \exp\left(-jk^2 (a(k)^2 + b(k)^2)\right) \right. \\ &\quad \left. + \frac{2i}{\sqrt{V}} \sum_n [\alpha_n (a(k) \cos(k \cdot x_n) - b(k) \sin(k \cdot x_n))] \right] \\ &\times \left[\prod_k' \int 2da(k) db(k) \exp\left(-jk^2 (a(k)^2 + b(k)^2)\right) \right]^{-1} \end{aligned} \quad (\text{C.5})$$

Here the prime indicates that the product is over k -values with $k_y > 0$. Using

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx) dx = \exp\left(\frac{b^2}{4a}\right) \sqrt{\frac{\pi}{a}} \quad (\text{C.6})$$

the integrals over $a(k)$ and $b(k)$ are easily performed. (C.5) simplifies to

$$\prod_k' \exp \left(-\frac{1}{Vjk^2} \left[\left(\sum_n \alpha_n \cos(k \cdot x_n) \right)^2 + \left(\sum_n \alpha_n \sin(k \cdot x_n) \right)^2 \right] \right) \quad (\text{C.7})$$

The product is evaluated by writing it as an exponentiated sum, and then replacing the sum by an integral. We thus have

$$\begin{aligned} & \prod_k' \exp \left(-\frac{1}{Vjk^2} \left[\left(\sum_n \alpha_n \cos(k \cdot x_n) \right)^2 + \left(\sum_n \alpha_n \sin(k \cdot x_n) \right)^2 \right] \right) \\ &= \exp \left(-\sum_k' \frac{1}{Vjk^2} \sum_{n,m} \alpha_n \alpha_m \cos(k \cdot (x_n - x_m)) \right) \end{aligned} \quad (\text{C.8})$$

Since the density of degrees of freedom for the field h^2 is $P^{(2)}(k)$ (see (3.15)) we can replace a sum by an integral as follows:

$$\sum_k \rightarrow V \int_k \frac{d^2 k}{(2\pi)^2} P^{(2)}(k) \quad (\text{C.9})$$

We can thus write the exponent of (C.8) as

$$-\frac{1}{2j} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2} P^{(2)}(k) \sum_{n,m} \alpha_n \alpha_m \cos(k \cdot (x_n - x_m)) \quad (\text{C.10})$$

The extra factor $\frac{1}{2}$ comes from taking into account the condition $k_y > 0$. Using (3.27) we can write this as

$$-\frac{1}{2} \sum_{n,m} \alpha_n \alpha_m G(x_n - x_m) \quad (\text{C.11})$$

from which it follows that

$$\begin{aligned} & \langle \exp(i \sum_n \alpha_n h(x_n)) \rangle \\ &= \exp(i \sum_n \alpha_n h^{(1)}(x_n)) \exp\left(-\frac{1}{2} \sum_{n,m} \alpha_n \alpha_m G(x_n - x_m)\right) \end{aligned} \quad (\text{C.12})$$

This proves (3.45) in this special case. Proving the general case is a matter of taking limits in (C.12). First we take the continuum limit in (C.12). Let T be a distribution generated by a bounded continuous function $T(x)$ i.e. $Th = \int d^2 x T(x) h(x)$. Then since there exist distributions T_n of the form (C.1), such that

$$\lim_{n \rightarrow \infty} T_n h = Th \quad (\text{C.13})$$

(3.45) is also valid in this case. Since for every distribution u there exist continuous functions $u_n(x)$ such that

$$\lim_{n \rightarrow \infty} u_n h = uh \quad (\text{C.14})$$

(see [31]), (3.45) follows.

Bibliography

- [1] Amit D J, Goldschmidt Y Y and Grinstein G 1980 J. Phys. A **13** 585.
- [2] Amit D J 1978 Field Theory, the Renormalization Group, and Critical Phenomena, World Scientific, Singapore.
- [3] Baxter R J 1970 Phys. Rev. B **1** 2199.
- [4] Baxter R J 1971 Stud. Appl. Math. (M.I.T.) **50** 51
- [5] Baxter R J 1972 Ann. Phys. **70** 193.
- [6] van Beijeren H 1977 Phys. Rev. Lett. **38** 993.
- [7] Binney J J, Dowrick N J, Fisher A J, and Newman M E J 1992 The Theory of Critical Phenomena, Oxford University Press, New York.
- [8] Fan C and Wu F Y 1970 Phys. Rev. B **2** 723.
- [9] Herring C 1953 Structure and Properties of Solid Surfaces, University Press, Chicago.
- [10] Hurst C A and Green H S 1960 J. Chem. Phys. **33** 1059.
- [11] Kadanoff L P 1979 Ann. Phys. (NY) **120** 39.
- [12] Kadanoff L P and Brown A C 1979 Ann. Phys. (NY) **121** 318.
- [13] Kasteleyn P W 1963 J. Math. Phys. **4** 287.
- [14] Kendal M G 1952 Advanced theory of statistics Vol. 1, Griffin, London.
- [15] Knops H J F 1980 Fundamental problems in statistical mechanics Vol. 5, North-holland Publishing Company
- [16] Knops H J F 1980 Ann. Phys. (NY) **128** 448.
- [17] Knops H J F 1982 Ann. Phys. (NY) **138** 155.
- [18] Kosterlitz M and Thouless D J 1973 J. Phys. **6** 1181.
- [19] Kosterlitz M and Thouless D J 1974 J. Phys. **7** 1046.
- [20] Landau L D 1965 Collected Papers, Pergamon, Oxford.
- [21] Lieb E H 1967 Phys. Rev. **162** 162.

- [22] Lieb E H 1967 Phys. Rev. Lett. **18** 1046.
- [23] Lieb E H 1967 Phys. Rev. Lett. **18** 108.
- [24] Luijten E, van Beijeren H, Blöte H W J 1994 Phys. Rev. Lett. **73** 456.
- [25] Ma S -K 1976 The Modern Theory of Critical Phenomena, Benjamin, Reading.
- [26] Montroll E W 1964 Combinatorial Mathematics, John Wiley & Sons, inc., New York.
- [27] den Nijs M 1979 J. Phys. A **12** 1857.
- [28] Niemeijer Th and van Leeuwen J M J 1976 Phase Transitions and Critical Phenomena, Vol. 6, Academic Press, London.
- [29] Ohta T and Jasnow D 1979 Phys. Rev. B **20** 139.
- [30] Parisi G 1988 Statistical Field Theory, Addison Wesley, Redwood City.
- [31] Rudin W 1973 Functional Analyses, Tata McGraw-Hill Publishing Company Ltd, New Delhi.
- [32] Schwartz L 1966 Théorie des distributions, Hermann & Cie, Paris.
- [33] Shenker S J, Kadanoff L P and Pruisken A M M 1979 J. Phys. A **12** 91.
- [34] Stanley H E 1971 Introduction to Phase Transitions and Critical Phenomena, Oxford University Press, Oxford.
- [35] Weeks J D 1980, Ordering in Strongly Fluctuating Condensed Matter Systems, Plenum, New York.
- [36] Wegner F J 1976 Phase Transitions and Critical Phenomena, Vol. 6, Academic Press, London.
- [37] Wilson K G and Kogut J B 1974 Phys. Rep. **12C** 77.
- [38] Wolf P E, Gallet F, Balibar S, Rolley E and Nozières P 1985 J. Physique **46** 1987.
- [39] Wu F Y and Lin K Y 1975 Phys. Rev. B **12** 419.
- [40] Youngblood R W and Axe J D 1980 Phys. Rev. B **21** 5212.
- [41] Zinn-justin J 1989 Quantum Field Theory and Critical Phenomena, Oxford University Press, Oxford.